Algebraic and geometric properties of minimal tensor rank decompositions

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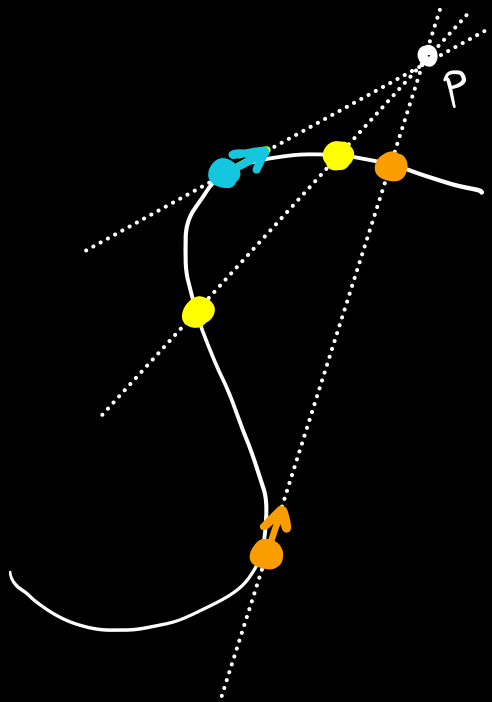
Ranks and zero-dimensional schemes

$X \subset \mathbb{P}^N$: projective irreducible variety

$p \in \mathbb{P}^N$



X-Rank

$$\mathrm{rk}_X(p) = \min_r \{ p \in \langle x_1, \ldots, x_r \rangle \ : \ x_i \in X \}$$

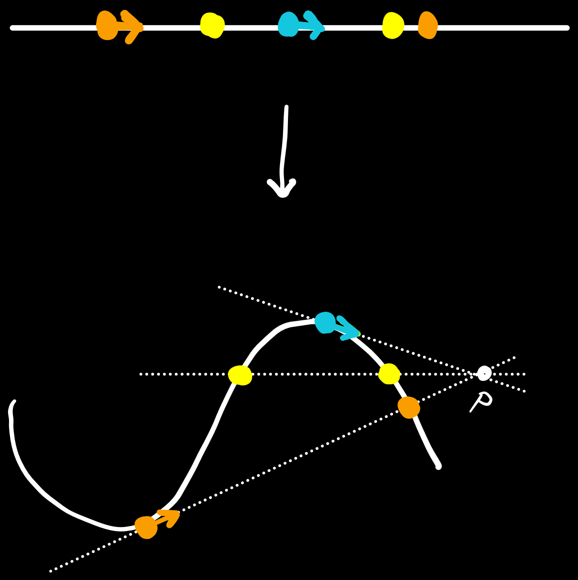
Cactus X-Rank

$$\mathrm{crk}_X(p) = \min_r \{ p \in \langle Z \rangle \ : \ Z \subset X, \ \ Z \text{ is 0-dim with } \mathbf{len}(Z) = r \}$$

Ranks and zero-dimensional schemes

$Y \subset \mathbb{P}^n$: projective irreducible variety $\mathscr{L}$: line bundle

$\phi_{\mathscr{L}} : Y \to \mathbb{P}(H^0(Y, \mathscr{L})^\vee) = \mathbb{P}^N$ $X = \phi_{\mathscr{L}}(Y) \subset \mathbb{P}^N$



X-Rank

$$\mathrm{rk}_X(p) = \min_r \{p \in \langle \phi_{\mathscr{L}}(x_1), \ldots, \phi_{\mathscr{L}}(x_r) \rangle \ : \ x_i \in Y\}$$

Cactus X-Rank

$$\mathrm{crk}_X(p) = \min_r \{p \in \langle \phi_{\mathscr{L}}(Z) \rangle \ : \ Z \subset Y, \ \ Z \text{ is 0-dim with } \mathbf{len}(Z) = r\}$$

Ranks and zero-dimensional schemes

$\nu_d(\mathbb{P}^n)$: degree-d Veronese variety

$$\nu_d : \mathbb{P}(\mathrm{Sym}^1\mathbb{C}^{n+1}) \to \mathbb{P}(\mathrm{Sym}^d\mathbb{C}^{n+1}), \quad [L] \to [L^d]$$

Rank

$$\mathrm{rk}(F) = \min_r \{F \in \langle L_1^d, \ldots, L_r^d \rangle \ : \ L_i \in \mathrm{Sym}^1\mathbb{C}^{n+1}\}$$

Cactus Rank

$$\mathrm{crk}(p) = \min_r \{F \in \langle \nu_d(Z) \rangle \ : \ Z \subset \mathbb{P}^n, \ \ Z \text{ is 0-dim with } \mathrm{len}(Z) = r\}$$

Ranks and zero-dimensional schemes

$\nu_{\underline{1}}(\mathbb{P}(\mathbb{C}^{\underline{n}}))$: Segre variety $\qquad \underline{n} = (n_1, \ldots, n_d)$

$\nu_{\underline{1}} : \mathbb{P}(\mathbb{C}^{\underline{n}}) \to \mathbb{P}(\mathbb{C}^{\otimes \underline{n}}), \quad ([v_1], \ldots, [v_d]) \to [v_1 \otimes \cdots \otimes v_d]$

Rank

$$\mathrm{rk}(T) = \min_r \{ F \in \langle \nu_{\underline{1}}(A_1), \ldots, \nu_{\underline{1}}(A_r) \rangle \ : \ A_i = (v_{i,1}, \ldots, v_{i,d}) \in \mathbb{P}(\mathbb{C}^{\underline{n}}) \}$$

Cactus Rank

$$\mathrm{crk}(p) = \min_r \{ F \in \langle \nu_{\underline{1}}(Z) \rangle \ : \ Z \subset \mathbb{P}(\mathbb{C}^{\underline{n}}), \ Z \text{ is 0-dim with } \mathbf{len}(Z) = r \}$$

Ranks and zero-dimensional schemes

$F \in \mathrm{Sym}^d \mathbb{C}^{n+1}$

$\mathrm{Ann}(F) = \{G \;:\; G \circ F = 0\}$

$\circ$ = action by derivation

Apolarity Lemma

[Iarrobino-Kanev, Gallet-Ranestad-Villamizar, Gałazka]

The following are equivalent:

i. $F \in \langle \nu_d(Z) \rangle$

ii. $I(Z) \subset \mathrm{Ann}(F)$

If so, we say that Z is **apolar** to F

We study rank / cactus rank / additive decompositions of F by looking at 0-dimensional schemes apolar to F

Ranks and zero-dimensional schemes

$F \in \mathrm{Sym}^d \mathbb{C}^{n+1}$

Questions

- What is the rank / cactus rank of F?
- Can we exhibit a minimal decomposition of F?
- **How do minimal decompositions looks like?
  I.e., how do minimal apolar schemes look like?**

Global properties: Varieties of Sums of Powers

$F \in \mathrm{Sym}^d \mathbb{C}^{n+1}$ $\qquad$ $r \in \mathbb{N}$

[Ranestad-Schreyer]

$$\mathrm{VSP}_r(F) = \overline{\{\{[L_1], \dots, [L_r]\} \ : \ F \in \langle L_1^d, \dots, L_r^d \rangle\}} \subset \mathrm{Hilb}_r(\mathbb{P}^n)$$

Examples

Considering generic forms:

- For $n = 2$, $\mathrm{VSP}_6(F)$ is a smooth Fano 3-fold of degree 22 [Mukai]
- For $n = 4$, $\mathrm{VSP}_8(F)$ is a smooth Fano 5-fold of degree 660 [Ranestad-Schreyer]

Note: cases corresponding to defective Veronese varieties

Global properties: Regularity and Hilbert Function

$F \in \mathrm{Sym}^d \mathbb{C}^{n+1}$

Question

How do **Hilbert functions** of apolar sets of points of minimal cardinality look like?

Which **regularities** can be attained?

Bernard ones asked

Can you provide an explicit example of F with apolar sets of points of minimal cardinality with different Hilbert functions and different regularities?

Global properties: Regularity and Hilbert Function

Binary Forms

[Sylvester]

$F \in \mathrm{Sym}^d \mathbb{C}^2$. Then, $\mathrm{Ann}(F) = (G_1, G_2)$
with $\deg(G_1) \leq \deg(G_2)$ and $\deg(G_1) + \deg(G_2) = \deg(F) + 2$

- If G_1 is square-free: $\mathrm{rk}(F) = \deg(G_1)$ and minimal decompositions are given by roots of G_1
- Otherwise, $\mathrm{rk}(F) = \deg(G_2)$ and minimal decompositions are given by roots of square-free forms $HG_1 + \lambda G_2$

$\deg(H) = \deg(G_2) - \deg(G_1), \lambda \in \mathbb{C}$

Global properties: Regularity and Hilbert Function

Monomials

[Carlini-Catalisano-Geramita, Buczynska-Buczynski-Teitler]

$M = x_0^{a_0} \cdots x_n^{a_n}$, $\quad a_0 \le a_1 \le \cdots \le a_n$. Then, $\mathrm{Ann}(M) = (x_0^{a_0+1}, \ldots, x_n^{a_n+1})$.

$$\mathrm{rk}(M) = \frac{1}{a_0+1} \prod_{i=1}^{n} (a_i + 1)$$

and

all minimal decompositions are complete intersections of degrees $(a_1+1, \ldots, a_n+1)$

Abuse of notation: $\mathrm{Ann}(M)$ and M lives in different polynomial rings which are "dual" with respect to the apolar action

Global properties: Regularity and Hilbert Function

Examples

[Angelini-Chiantini-Oneto]

We construct

- Ternary form of degree 10 and rank 22 which admits different decompositions with different Hilbert function

  Note: 22 is the generic rank
- Ternary form of degree 13 and rank 30 which admits different decompositions with different regularity

  Note: 30 is subgeneric rank

Tools

[Angelini-Chiantini, Angelini-Chiantini-Vannieuwenhoven]

Liaison Theory and *Cayley-Bacharach properties*

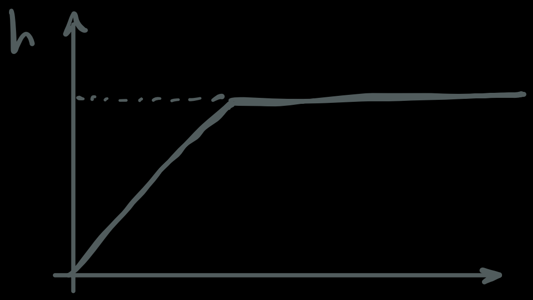
They provide identifiability criteria for specific tensors

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

1. A : set of 12 general points in $\mathbb{P}^2$



	0	1	2	3	4	5	
h_A :	1	3	6	9	12	12	$\cdots$
Δh_A :	1	2	3	3	3	–	

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

1. A : set of 12 general points in $\mathbb{P}^2$
2. Link A through a complete intersection $X = A \cup Z_1$ of type (6,7)

	0	1	2	3	4	5	6	7	8	9	10	11
Δh_X :	1	2	3	4	5	6	6	5	4	3	2	1
Δh_A :								3	3	3	2	1
Δh_{Z_1} :	1	2	3	4	5	6	6	2	1			

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

1. A : set of 12 general points in $\mathbb{P}^2$
2. Link A through a complete intersection $X = A \cup Z_1$ of type (6,7)
3. Link Z_1 through a complete intersection $Y = Z_1 \cup Z_2$ of type (6,10)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Δh_y :	1	2	3	4	5	6	6	6	6	6	5	4	3	2	1
Δh_{Z_1} :							1	2	6	6	5	4	3	2	1
Δh_{Z_2} :	1	2	3	4	5	6	5	4							

Global properties: Regularity and Hilbert Function

[Angelini-Chiantini]

$$\dim\langle \nu_d(Z_1)\rangle \cap \langle \nu_d(Z_2)\rangle = \mathrm{len}(Z_1 \cap Z_2) - 1 + h^1_{Z_1 \cup Z_2}(d)$$

$$h^1_Z(d) = \mathrm{len}(Z) - h_Z(d) = \sum_{j=d+1}^{\infty} \Delta h_Z(i)$$

Example 1 [Angelini-Chiantini-Oneto]

1. A : set of 12 general points in $\mathbb{P}^2$
2. Link A through a complete intersection $X = A \cup$ ... of type (6,7)
3. Link Z_1 through a complete intersection $Y = Z_1 \cup Z_2$ of type (6,10)
4. There exists a degree-13 polynomial $[F] \in \langle \nu_{13}(Z_1)\rangle \cap \langle \nu_{13}(Z_2)\rangle$

We need to show that it has indeed rank equal to 30

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Δh_Y :	1	2	3	4	5	6	6	6	6	6	5	4	3	2	1
Δh_{Z_1} :							1	2	6	6	5	4	3	2	1
Δh_{Z_2} :	1	2	3	4	5	6	5	4							

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

Assume that there is a scheme Z' of $\mathrm{len}(Z') \leq 29$ apolar to the same F

Assume first that $Z' \cap Z_1 = \varnothing$

$\Delta h_{Z' \cup Z_1}$:

Δh_{Z_1} :	1	2	3	4	5	6	6	2	1
$\Delta h_{Z'}$:									

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

Assume that there is a scheme Z' of $\mathrm{len}(Z') \leq 29$ apolar to the same F

Assume first that $Z' \cap Z_1 = \emptyset$

Up to degree 5, we have maximal growth; i.e., $\sum_{i=0}^{5} \Delta h_{Z' \cup Z_1}(i) \geq \sum_{i=0}^{5} \Delta h_{Z_1}(i) = 21$

$\Delta h_{Z' \cup Z_1}$:	1	2	3	4	5	6			
Δh_{Z_1} :	1	2	3	4	5	6	6	2	1
$\Delta h_{Z'}$:									

Cayley-Bacharach property $CB(d)$: $\forall p \in Z \, \forall F \in I(Z \setminus p)_d \quad \Rightarrow \quad F \in I(Z)_d$

Proposition [Angelini-Chiantini

If A non-redundant and apolar to $F \in \mathrm{Sym}^d(\mathbb{C}^{n+1})$
and B apolar to F such that $A \cap B = \emptyset$, then $Z = A \cup B$ satisfies $CB(d)$.

$$CB(d) \quad \Rightarrow \quad \sum_{j=0}^{i} \Delta h_Z(j) \geq \sum_{j=0}^{i} \Delta h_Z(d+1-j)$$

Z' satisfies Cayley-Bacharach property $CB(13)$ [Angelini-Chiantini]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\Delta h_{Z' \cup Z_1}$:	1	2	3	4	5	6									≥ 1
Δh_{Z_1} :	1	2	3	4	5	6	6	2	1						
$\Delta h_{Z'}$:															

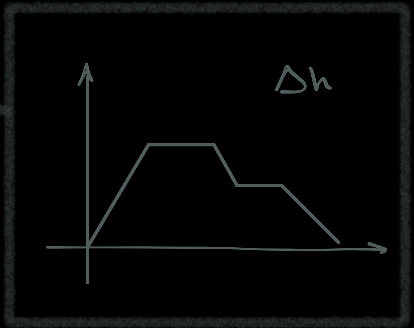
Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

Assume that there is a scheme Z' of $\mathrm{len}(Z') \leq 29$ apolar to the same F

Assume first that $Z' \cap Z_1 = \varnothing$



We cannot have plateau below 6 [Davis]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\Delta h_{Z' \cup Z_1}$:	1	2	3	4	5	6	≥ 6	≥ 6	≥ 6	≥ 6	≥ 5	≥ 4	≥ 3	≥ 2	≥ 1
Δh_{Z_1} :	1	2	3	4	5	6	6	2	1						
$\Delta h_{Z'}$:															

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

Assume that there is a scheme Z' of $\mathrm{len}(Z') \leq 29$ apolar to the same F

Assume first that $Z' \cap Z_1 = \emptyset$

Contradiction: $\mathrm{len}(Z') \geq 30$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\Delta h_{Z' \cup Z_1}$:	1	2	3	4	5	6	≥ 6	≥ 6	≥ 6	≥ 6	≥ 5	≥ 4	≥ 3	≥ 2	≥ 1
Δh_{Z_1} :	1	2	3	4	5	6	6	2	1						
$\Delta h_{Z'}$:								≥ 4	≥ 5	≥ 6	≥ 5	≥ 4	≥ 3	≥ 2	≥ 1

Global properties: Regularity and Hilbert Function

Example 1

[Angelini-Chiantini-Oneto]

Assume that there is a scheme Z' of $\mathrm{len}(Z') \leq 29$ apolar to the same F

If $Z' \cap Z_1 \neq \emptyset$, then we can reduce to the previous case:

$$Z_1 = \{L_1, \dots, L_{30}\} \qquad Z' = \{L_1, \dots, L_k, M_{k+1}, \dots, M_{30}\}$$

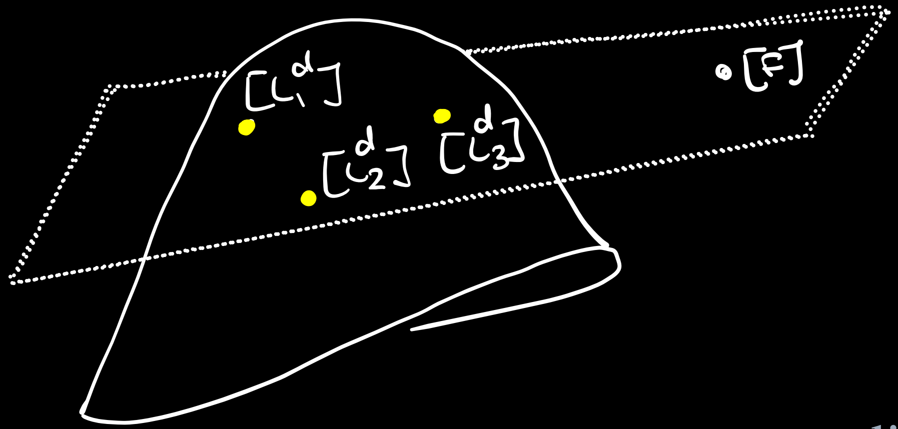
$$F = \sum_{i=1}^{30} a_i L_i^{13} = \sum_{i=1}^{k} b_i L_i^{13} + \sum_{i=k+1}^{30} c_i M_i^{13}$$

Then, we can replace Z' with $Z' \smallsetminus (Z_1 \cap Z')$

Global properties: Regularity and Hilbert Function

Trivial Remark

If Z is a *minimal* set of reduced points spanning $F \in \mathbf{Sym}^d(\mathbb{C}^{n+1})$, then it is regular in degree d



(affine dimension)

$$\dim\langle \nu_d(Z)\rangle = h_Z(d)$$

If Z is not d-regular, then $\{[L_1^d], \ldots, [L_r^d]\}$are linearly independent

Global properties: Regularity and Hilbert Function

Trivial Remark

If Z is a *minimal* set of reduced points spanning $F \in \mathrm{Sym}^d(\mathbb{C}^{n+1})$, then it is regular in degree d

Question

[Bernardi-Taufer, Bernardi-Oneto-Taufer]

Given $F \in \mathrm{Sym}^d\mathbb{C}^{n+1}$, can we bound the regularity of any *minimal* apolar schemes?

In particular, **can we prove that it is smaller than the degree d of the polynomial?**

note: this question is relevant to understand the complexity of decomposition algorithms

Remark

Irredundant (i.e., minimal by inclusion) instead of *minimal length* is not enough!

Global properties: Regularity and Hilbert Function

Example

[Bernardi-Oneto-Taufer]

$F = x_0G_1 + x_1G_2$

with
$$G_1 = 10x_0^3 - 4x_0^2x_1 + 4x_0^2x_2 - 4x_0x_1^2 - 8x_0x_1x_2 - 3x_0x_2^2 - 8x_1^3 - 4x_2^3,$$
$$G_2 = 5x_0^3 + 9x_0x_1^2 - 5x_1^3 - 7x_1^2x_2 + 6x_1x_2^2 - x_2^3\,.$$

The 0-dimensional scheme evincing this decomposition has Hilbert function

$$1 \quad 3 \quad 6 \quad 10 \quad 11 \quad 12 \quad 12 \quad \cdots$$

Given $F = L^{d-k}G$ we consider the scheme defined by $\mathrm{Ann}(g)$ where $g = G|_{L=1}$ regarded as a projective 0-dimensional scheme supported at $[L]$. **This scheme is apolar to F**

Given $F = \sum_{i=1}^{r} L_i^{d-k_i}G_i$, we construct an **apolar scheme** summand by summand

Global properties: Regularity and Hilbert Function

Example

[Bernardi-Oneto-Taufer]

$$F = x_0 G_1 + x_1 G_2$$

with
$$G_1 = 10x_0^3 - 4x_0^2x_1 + 4x_0^2x_2 - 4x_0x_1^2 - 8x_0x_1x_2 - 3x_0x_2^2 - 8x_1^3 - 4x_2^3,$$
$$G_2 = 5x_0^3 + 9x_0x_1^2 - 5x_1^3 - 7x_1^2x_2 + 6x_1x_2^2 - x_2^3.$$

The 0-dimensional scheme evincing this decomposition has Hilbert function

$$1 \quad 3 \quad 6 \quad 10 \quad 11 \quad 12 \quad 12 \quad \cdots$$

but it is irredundant:

- If there was, it should correspond to a decomposition $F = X_0 Q_1 + X_1 Q_2$

but then, $X_0(G_1 - Q_1) = X_1(G_2 - Q_2)$

Global properties: Regularity and Hilbert Function

Example

[Bernardi-Oneto-Taufer]

$F = x_0 G_1 + x_1 G_2$

with
$$G_1 = 10x_0^3 - 4x_0^2x_1 + 4x_0^2x_2 - 4x_0x_1^2 - 8x_0x_1x_2 - 3x_0x_2^2 - 8x_1^3 - 4x_2^3,$$
$$G_2 = 5x_0^3 + 9x_0x_1^2 - 5x_1^3 - 7x_1^2x_2 + 6x_1x_2^2 - x_2^3\,.$$

The 0-dimensional scheme evincing this decomposition has Hilbert function

$$1 \quad 3 \quad 6 \quad 10 \quad 11 \quad 12 \quad 12 \quad \cdots$$

but it is irredundant:

- If there was, it should correspond to a decomposition $F = X_0(G_1 - X_1T) + X_1(G_2 + X_0T)$
- We checked computationally that it has to be $T = \lambda X_0 X_1$

Global properties: Regularity and Hilbert Function

Example

[Bernardi-Oneto-Taufer]

$$F = x_0 G_1 + x_1 G_2$$

with
$$\begin{aligned} G_1 &= 10x_0^3 - 4x_0^2x_1 + 4x_0^2x_2 - 4x_0x_1^2 - 8x_0x_1x_2 - 3x_0x_2^2 - 8x_1^3 - 4x_2^3, \\ G_2 &= 5x_0^3 + 9x_0x_1^2 - 5x_1^3 - 7x_1^2x_2 + 6x_1x_2^2 - x_2^3 . \end{aligned}$$

The 0-dimensional scheme evincing this decomposition has Hilbert function

$$1 \quad 3 \quad 6 \quad 10 \quad 11 \quad 12 \quad 12 \quad \cdots$$

but it is irredundant:

- If there was, it should correspond to a decomposition $F = X_0(G_1 - X_1T) + X_1(G_2 + X_0T)$
- We checked computationally that it has to be $T = \lambda X_0 X_1$
- All such decompositions are evinced by the same 0-dimensional scheme

Local properties: Decomposition Loci

Question

Given $F \in \mathrm{Sym}^d\mathbb{C}^{n+1}$, which linear forms may appear in a minimal decomposition?

Given $T \in \mathbb{C}^{\underline{n}}$, which rank-one tensors may appear in a minimal decomposition?

TensorGame - a reinforcement learning approach

[Fawzi et al.]

- Start with $T_0 := T$
- At each step *t*, the player takes a rank-one tensor $v_1 \otimes \cdots \otimes v_d$ and substitutes $T_t := T_{t-1} - v_1 \otimes \cdots \otimes v_d$
- The game ends when $T_t = 0$.

Challenge. Find the optimal way, i.e., conclude in a number of steps equal to $\mathrm{rk}(T)$

Local properties: Decomposition Loci

Question

Given $F \in \mathrm{Sym}^d\mathbb{C}^{n+1}$, which linear forms may appear in a minimal decomposition?

[Carlini-Catalisano-Oneto]

The **decomposition locus** of F is

$$\mathscr{D}(F) = \{[L] \in \mathbb{P}\mathrm{Sym}^1(\mathbb{C}^{n+1}) \ : \ \mathrm{rk}(F - \lambda L^d) = \mathrm{rk}(F) - 1 \text{ for some } \lambda\}$$

The **forbidden locus** of F is the complement of the decomposition locus

$$\mathscr{F}(F) = \mathbb{P}(\mathrm{Sym}^1(\mathbb{C}^{n+1})) \smallsetminus \mathscr{D}(F)$$

note: we always look at them in the concise space

Local properties: Decomposition Loci

Question

Given $T \in \mathbb{C}^{\underline{n}}$, which rank-one tensors may appear in a minimal decomposition?

[Bernardi-Oneto-Santarsiero]

The **decomposition locus** of T is

$$\mathcal{D}(T) = \{([v_1], \dots, [v_d]) \in \mathbb{P}(\mathbb{C}^{\underline{n}}) \ : \ \mathrm{rk}(T - \lambda v_1 \otimes \cdots \otimes v_d) = \mathrm{rk}(T) - 1 \text{ for some } \lambda\}$$

The **forbidden locus** of T is the complement of the decomposition locus

$$\mathcal{F}(T) = \mathbb{P}(\mathbb{C}^{\underline{n}}) \setminus \mathcal{D}(T)$$

note: we always look at them in the concise space

Local properties: Decomposition Loci

[Carlini-Catalisano-Oneto]

We compute the decomposition / forbidden loci of homogeneous polynomials such as: binary forms, monomials, plane cubics, other families of polynomials

Local properties: Decomposition Loci

[Carlini-Catalisano-Oneto]

We compute the decomposition / forbidden loci of homogeneous polynomials such as: binary forms, monomials, plane cubics, other families of polynomials

[Mourrain-Oneto]

We perform the TensorGame for homogeneous polynomials of rank at most 5
In particular, we get a classification of homogeneous polynomials of rank at most 5 in terms of algebraic properties of apolar ideals
and we describe how to construct a minimal decomposition

Local properties: Decomposition Loci

Binary Forms

[Carlini-Catalisano-Oneto]

$F \in \mathrm{Sym}^d\mathbb{C}^2$. Then, $\mathrm{Ann}(F) = (G_1, G_2)$

with $\deg(G_1) \leq \deg(G_2)$ and $\deg(G_1) + \deg(G_2) = \deg(F) + 2$

- $\mathrm{rk}(F) < \lceil (d+1)/2 \rceil$ then $\mathscr{D}(F) =$ set of roots of G_1
- $\mathrm{rk}(F) > \lceil (d+1)/2 \rceil$ then $\mathscr{F}(F) =$ set of roots of G_1
- $\mathrm{rk}(F) = \lceil (d+1)/2 \rceil$ and d is odd: then $\mathscr{D}(F) =$ set of roots of G_1
- $\mathrm{rk}(F) = \lceil (d+1)/2 \rceil$ and d is even: then $\mathscr{F}(F) =$ non-empty finite set of points

Local properties: Decomposition Loci

Binary Forms

[Carlini-Catalisano-Oneto]

$F \in \mathrm{Sym}^d\mathbb{C}^2$. Then, $\mathrm{Ann}(F) = (G_1, G_2)$
with $\deg(G_1) \leq \deg(G_2)$ and $\deg(G_1) + \deg(G_2) = \deg(F) + 2$

- $\mathrm{rk}(F) < \lceil (d+1)/2 \rceil$ then $\mathscr{D}(F) =$ set of roots of G_1

This is the case G_1 is square-free and it is the only generator of its degree

Local properties: Decomposition Loci

Binary Forms

[Carlini-Catalisano-Oneto]

$F \in \mathrm{Sym}^d\mathbb{C}^2$. Then, $\mathrm{Ann}(F) = (G_1, G_2)$

with $\deg(G_1) \leq \deg(G_2)$ and $\deg(G_1) + \deg(G_2) = \deg(F) + 2$

- $\mathrm{rk}(F) > \lceil (d+1)/2 \rceil$ then $\mathcal{F}(F) =$ set of roots of G_1

In this case, G_1 is not square-free. Let L be any factor of G_1.

G_1, G_2 have no common factors because $\mathrm{Ann}(F)$ is Gorenstein in codimesion 2

hence, all polynomials of $\mathrm{Ann}(F)$ divisible by L are multiples of G_1 and then not square-free

Local properties: Decomposition Loci

Monomials

[Carlini-Catalisano-Geramita]

$M = x_0^{a_0} \cdots x_n^{a_n}, \quad a_0 \leq a_1 \leq \cdots \leq a_n$. Then, $\mathrm{Ann}(M) = (x_0^{a_0+1}, \ldots, x_n^{a_n+1})$.

$$\mathrm{rk}(M) = \frac{1}{a_0+1} \prod_{i=1}^{n} (a_i + 1)$$

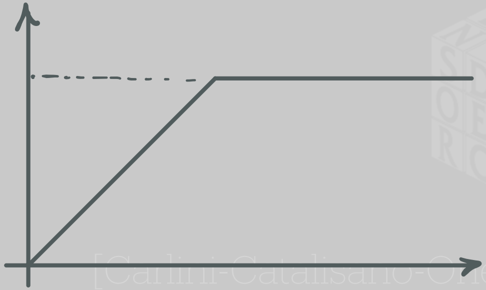
Idea of proof. Assume Z apolar and minimal (i.e., computes the rank) for M.

Consider $Z' := Z \setminus (Z \cap \{x_0 = 0\})$. Then, x_0 is a non-zero divisor for $I(Z)$.

$$I(Z') + (x_0) = I(Z) : (x_0) + (x_0) \subset \mathrm{Ann}(M) : (x_0) + (x_0) = (x_0, x_1^{a_1+1}, \ldots, x_n^{a_n+1})$$

Local properties: Decomposition Loci

- Given a zero-dimensional scheme $Z \subset \mathbb{P}^n$,
  the Hilbert function is strictly increasing until it reaches $\mathrm{len}(Z)$



Monomials

[Carlini-Catalisano-Oneto]

- If L is a non-zero divisor for Z, i.e., $Z \cap \{L = 0\} = \emptyset$, then
  the first difference of the Hilbert function of Z is equal to the Hilbert function of the quotient of $I(Z) + (L)$

$$\#(Z) = \sum_{i=0}^{\infty} \Delta h_Z(i) = \dim \mathbb{C}[x_0, \ldots, x_n]/I(Z) + (L)$$

Idea of proof. Assume Z apolar and minimal (i.e., computes the rank) for M.
Consider $Z' := Z \setminus (Z \cap \{x_0 = 0\})$. Then, x_0 is a non-zero divisor for $I(Z)$.

$$I(Z') + (x_0) = I(Z) : (x_0) + (x_0) \subset \mathrm{Ann}(M) : (x_0) + (x_0) = (x_0, x_1^{a_1+1}, \ldots, x_n^{a_n+1})$$

Then, from basic properties of Hilbert functions of 0-dimensional schemes:

$$\#(Z') \geq \dim \mathbb{C}[x_0, \ldots, x_n]/(x_0, x_1^{a_1+1}, \ldots, x_n^{a_n+1}) = (a_1 + 1)\cdots(a_n + 1)$$

Local properties: Decomposition Loci

Monomials

[Carlini-Catalisano-Geramita]

$M = x_0^{a_0} \cdots x_n^{a_n}$, $\quad a_0 \leq a_1 \leq \cdots \leq a_n$. Then, $\mathrm{Ann}(M) = (x_0^{a_0+1}, \ldots, x_n^{a_n+1})$.

$$\mathrm{rk}(M) = \frac{1}{a_0+1} \prod_{i=1}^{n} (a_i + 1)$$

Idea of proof. Assume Z apolar and minimal (i.e., computes the rank) for M.

Consider $Z' := Z \smallsetminus (Z \cap \{x_0 = 0\})$. Then, x_0 is a non-zero divisor for $I(Z)$.

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In particular: $Z = Z'$, namely **the hyperplane $\{x_0 = 0\}$ is forbidden**.

Local properties: Decomposition Loci

Monomials

[Carlini-Catalisano-Oneto]

$M = x_0^{a_0} \cdots x_n^{a_n}$, $\quad a_0 = a_1 = \ldots = a_m < a_{m+1} \leq \cdots \leq a_n$. Then,

$$\mathcal{F}(M) = \{x_0 \cdots x_m = 0\}$$

Idea of proof. Assume Z apolar and minimal (i.e., computes the rank) for M.

Consider $Z' := Z \setminus (Z \cap \{x_0 = 0\})$. Then, x_0 is a non-zero divisor for $I(Z)$.

$$I(Z') + (x_0) = I(Z) : (x_0) + (x_0) \subset \mathrm{Ann}(M) : (x_0) + (x_0) = (x_0, x_1^{a_1+1}, \ldots, x_n^{a_n+1})$$

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In particular: $Z = Z'$, namely **the hyperplane** $\{x_0 = 0\}$ **is forbidden**.

Local properties: Decomposition Loci

[Bernardi-Oneto-Santarsiero]

We compute the decomposition / forbidden loci of:

- matrices
- tangential tensors
- tensors which are in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$ by explicit computations from the 26 normal forms

(finite number of orbits wrt $\mathbf{GL} \times \mathbf{GL} \times \mathbf{GL}$)

Local properties: Decomposition Loci

Tangential Tensors

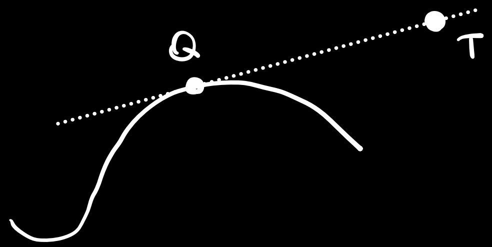
[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

is a tensor lying on the tangent space to the Segre variety at $Q = [v_1 \otimes \cdots \otimes v_d]$.

Then,

$$\mathcal{F}(T) = \{[Q]\}$$



Note: for the symmetric case, this follows from the previous examples since

$T = xy^{d-1}$ belongs to the tangent to the Veronese variety at $Q = [y^d]$

and indeed the forbidden locus is given by $\{x = 0\}$

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathscr{F}(T) = \{[Q]\}$$

Proof.

Step 1. $[Q] \in \mathscr{F}(T)$.

$T - \lambda Q$ is still tangential and concise for all $\lambda \neq 0$, hence it has the same rank.

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathscr{F}(T) = \{[Q]\}$$

Proof.

Step 2. The result holds for $d = 3$.

T has rank equal to 4.

The variety of rank-3 tensors in $(\mathbb{C}^2)^{\otimes 3}$ is defined by the Cayley hyperdeterminant.

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathcal{F}(T) = \{[Q]\}$$

Proof.

Step 3. If $P = p_1 \otimes \cdots \otimes p_d$ with $[p_i] \neq [v_i]$, then $P \notin \mathcal{F}(T)$.

Without loss of generalities: $v_i = (1 : 0)$ and $p_i = (p_i : 1)$

Consider the curve $f = f_1 \times \cdots \times f_d : \mathbb{P}^1 \to (\mathbb{P}^1)^{\times d}$, $f_i : (x : y) \mapsto (x + p_i y : y)$

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathscr{F}(T) = \{[Q]\}$$

Proof.

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Without loss of generalities: $v_i = (1 : 0)$ and $p_i = (p_i : 1)$

Consider the curve $f = f_1 \times \cdots \times f_d : \mathbb{P}^1 \to (\mathbb{P}^1)^{\times d}, \;\; f_i : (x : y) \mapsto (x + p_i y : y)$

It is a degree-d Rational Normal Curve passing through Q whose tangent line at Q contains T and passing through P. We conclude by the previous results on binary forms or monomials.

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathcal{F}(T) = \{[Q]\}$$

Proof.

Step 4. If $P = p_1 \otimes \cdots \otimes p_d$ with $[p_i] = [v_i]$ for $i = 1,\ldots,m$, $(m < d)$

If $d - m \geq 3$:

$$T + \lambda P = \sum_{i=1}^{m} v_1 \otimes \cdots w_i \cdots \otimes v_d + (v_1 \otimes \cdots \otimes v_m) \otimes T'$$

$T' = \sum_{i=m+1}^{d} v_{m+1} \otimes \cdots w_i \cdots \otimes v_d + \lambda p_{m+1} \otimes \cdots \otimes p_d$ which satisfies the assumptions of Step 3.

Local properties: Decomposition Loci

Tangential Tensors

[Bernardi-Oneto-Santarsiero]

$$T = v_1 \otimes v_2 \otimes \cdots \otimes w_d + \ldots + v_1 \otimes w_2 \otimes \cdots \otimes v_d + w_1 \otimes v_2 \otimes \cdots \otimes v_d$$

$$\mathscr{F}(T) = \{[Q]\}$$

Proof.

Step 4. If $P = p_1 \otimes \cdots \otimes p_d$ with $[p_i] = [v_i]$ for $i = 1,\ldots,m$, $(m < d)$

If $d - m < 3$:

$$T + \lambda P = \sum_{i=1}^{d-3} v_1 \otimes \cdots w_i \cdots \otimes v_d + (v_1 \otimes \cdots \otimes v_{d-3}) \otimes T'$$

$T' = \sum_{i=d-2}^{d} v_{d-2} \otimes \cdots w_i \cdots \otimes v_d + \lambda p_{d-2} \otimes p_{d-1} \otimes p_d$ which satisfies the assumptions of Step 2.


Thank you for the attention
Merci Bernard
et bon anniversaire!