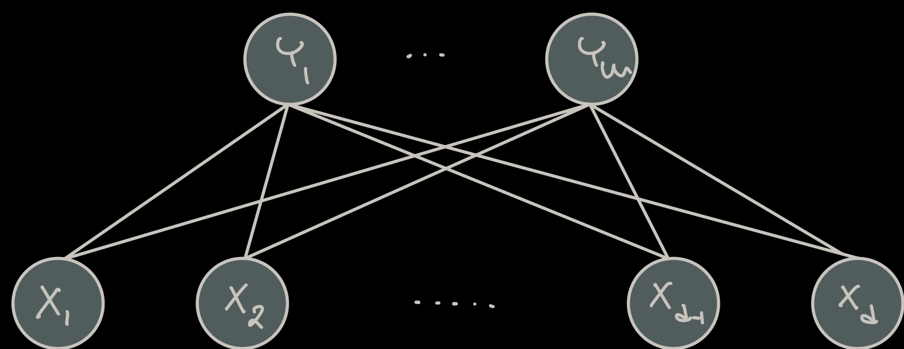
Hadamard-Hitchcock Decompositions

Alessandro ONETO

joint work with Nick VANNIEUWENHOVEN (KU Leuven)



UNIVERSITÀ DI TRENTO

TENORS
Tensor modEliNg, geOmetRy and optimiSation
Marie Skłodowska-Curie Doctoral Network
2024-2027

Tensors are nowadays ubiquitous in many domains of applied mathematics, computer science, signal processing, data processing, machine learning and in the emerging area of quantum computing. TENORS aims at fostering cutting-edge research in tensor sciences, stimulating interdisciplinary and intersectoriality knowledge developments between algebraists, geometers, computer scientists, numerical analysts, data analysts, physicists, quantum scientists, and industrial actors facing real-life tensor-based problems.

Partners:

1. Inria, Sophia Antipolis, France (B. Mourrain, A. Mantzaflaris)
2. CNRS, LAAS, Toulouse, France (D. Henrion, V. Magron, M. Skomra)
3. NWO-I/CWI, Amsterdam, the Netherlands (M. Laurent)
4. Univ. Konstanz, Germany (M. Schweighofer, S. Kuhlmann, M. Michałek)
5. MPI, Leipzig, Germany (B. Sturmfels, S. Telen)
6. Univ. Tromsoe, Norway (C. Riener, C. Bordin, H. Munthe-Kaas)
7. Univ. degli Studi di Firenze, Italy (G. Ottaviani)
8. Univ. degli Studi di Trento, Italy (A. Bernardi, A. Oneto, I. Carusotto)
9. CTU, Prague, Czech Republic (J. Marecek)
10. ICFO, Barcelona, Spain (A. Acin)
11. Artelys SA, Paris, France (M. Gabay)

Associate partners:

1. Quandela, France
2. Cambridge Quantum Computing, UK.
3. Bluetensor, Italy.
4. Arva AS, Norway.
5. HSBC Lab., London, UK.

15 PhD positions (2024-2027)

(recruitment expected around Oct. 2024)

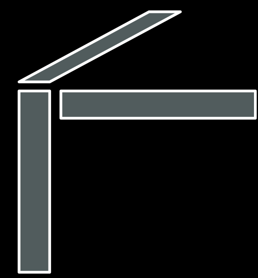
Scientific coord: B. Mourrain
Adm. manager: Linh Nguyen

Hitchcock Decompositions

(aka Canonical Polyadic (CP) or PARAFAC)

Let $T \in V_1 \otimes \cdots \otimes V_d$ $\qquad V_i$ – $\mathbb{k}$-vector spaces

rank-one tensor



$$T = v_1 \otimes \cdots \otimes v_d \quad \text{with} \quad v_i \in V_i$$

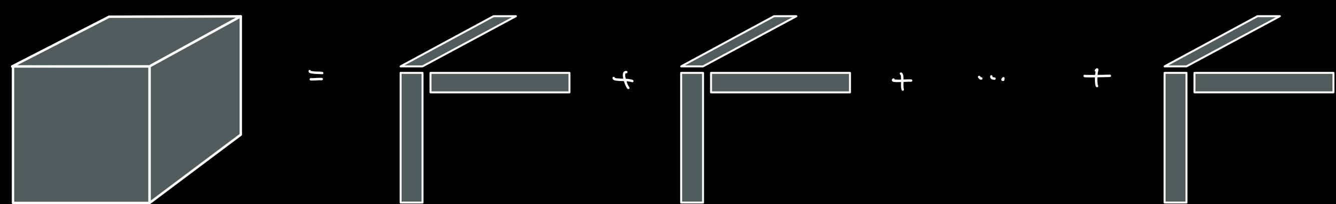
$$T_{i_1 \ldots i_d} = (v_1)_{i_1} \cdots (v_d)_{i_d} \quad \forall\, (i_1 \ldots i_d)$$

Hitchcock Decomposition

an expression of T as linear combination of rank-one tensors

$$T = \sum_{i=1}^{r} \lambda_i \, v_{i,1} \otimes \cdots \otimes v_{i,d} \qquad \lambda_i \in \mathbb{k}, \; v_{i,j} \in V_j$$

rank of T : smallest length of such decomposition



Mixtures of Independence Models

Let $\mathcal{X} = \{ X_1 \dots X_d \}$ discrete random variables,

$X_i \in [n_i] = \{ 1 \dots n_i \}$

$$T_{\mathcal{X}} \in \mathbb{R}^{n_1} \otimes \dots \otimes \mathbb{R}^{n_d} , \quad [T_{\mathcal{X}}]_{i_1 \dots i_d} = \mathbb{P}(X_1 = i_1, \dots, X_d = i_d)$$

Mixtures of Independence Models

Let $\mathcal{X} = \{X_1 \dots X_d\}$ discrete random variables,

$X_i \in [n_i] = \{1 \dots n_i\}$

$T_{\mathcal{X}} \in \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d}$, $[T_{\mathcal{X}}]_{i_1 \dots i_d} = \mathbb{P}(X_1 = i_1, \dots, X_d = i_d)$

$\mathcal{X}$ independent $=$ $T_{\mathcal{X}}$ rank-one tensor

$$\mathbb{P}(\mathcal{X} = \underline{i}) = \mathbb{P}(X_1 = i_1) \cdots \mathbb{P}(X_d = i_d)$$

$T_{\mathcal{X}} = v_1 \otimes \cdots \otimes v_d$ where $v_i = (\mathbb{P}(X_i = 1), \dots, \mathbb{P}(X_i = n_i))$

$\underline{n} = (n_1 \dots n_d)$

$\underline{i} = (i_1 \dots i_d) \in [\underline{n}] = [n_1] \times \cdots \times [n_d]$

$\mathcal{X} = \underline{i}$ means $X_j = i_j \ \forall j \in [d]$

Mixtures of Independence Models

Let $\mathcal{X} = \{ X_1 \dots X_d \}$ discrete random variables,

$$X_i \in [n_i] = \{1 \dots n_i\}$$

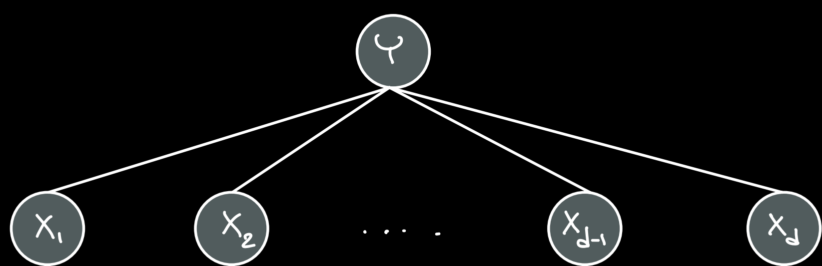
$$T_{\mathcal{X}} \in \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d} \, , \quad [T_{\mathcal{X}}]_{i_1 \dots i_d} = \mathbb{P}(X_1 = i_1, \dots, X_d = i_d)$$

$\mathcal{X}$ independent conditional to $Y \in [r]$ = $T_{\mathcal{X}}$ rank-r tensor

$$\mathbb{P}(\mathcal{X} = \underline{i}) = \sum_{j=1}^{r} \mathbb{P}(Y = j) \, \mathbb{P}(\mathcal{X} = \underline{i} \mid Y = r)$$

$$T_{\mathcal{X}|Y} = \sum_{j=1}^{r} \lambda_j \, v_{j,1} \otimes \cdots \otimes v_{j,d}$$

where $v_{j,i} = \big(\mathbb{P}(X_i = 1 \mid Y = j), \dots, \mathbb{P}(X_i = n_i \mid Y = j)\big)$



HADAMARD-HITCHCOCK DECOMPOSITIONS

Hadamard product

$T, S \in V_1 \otimes \cdots \otimes V_d$

$$[T * S]_{\underline{i}} = [T]_{\underline{i}} [S]_{\underline{i}}$$

$(r_1 \ldots r_m)$ - Hadamard-Hitchcock Decomposition $(\underline{r}\text{-HHD})$

$$T = T_1 * \cdots * T_m \qquad \text{where } \operatorname{rank}(T_i) = r_i$$

PRODUCTS OF MIXTURES OF INDEPENDENCE MODELS
(aka RESTRICTED BOLTZMANN MACHINES)

Let $\mathcal{X} = \{X_1 \dots X_d\}$ and $\mathcal{Y} = \{Y_1 \dots Y_m\}$

discrete random variables, $X_i \in [n_i]$ and $Y_i \in [r_i]$

such that $\mathcal{X}$ is independent conditionally to $\mathcal{Y}$

and viceversa.

$$\mathbb{P}(\mathcal{X} = \underline{i}) = \sum_{\underline{j} \in [\underline{r}]} \mathbb{P}(\mathcal{Y} = \underline{j})\, \mathbb{P}(\mathcal{X} = \underline{i} \mid \mathcal{Y} = \underline{j})$$

PRODUCTS OF MIXTURES OF INDEPENDENCE MODELS
(aka RESTRICTED BOLTZMANN MACHINES)

Let $\mathcal{X} = \{X_1 \dots X_d\}$ and $\mathcal{Y} = \{Y_1 \dots Y_m\}$

discrete random variables, $X_i \in [n_i]$ and $Y_i \in [r_i]$

such that $\mathcal{X}$ is independent conditionally to $\mathcal{Y}$

and viceversa.

$$\mathbb{P}(\mathcal{X} = \underline{i}) = \sum_{\underline{j} \in [\underline{r}]} \mathbb{P}(\mathcal{Y} = \underline{j})\, \mathbb{P}(\mathcal{X} = \underline{i} \mid \mathcal{Y} = \underline{j}) =$$

$$= \sum_{\underline{j} \in [\underline{r}]} \mathbb{P}(\mathcal{Y} = \underline{j})\, \mathbb{P}(X_1 = i_1 \mid \mathcal{Y} = \underline{j}) \cdots \mathbb{P}(X_d = i_d \mid \mathcal{Y} = \underline{j})$$

PRODUCTS OF MIXTURES OF INDEPENDENCE MODELS
(aka RESTRICTED BOLTZMANN MACHINES)

Let $\mathcal{X} = \{ X_1 \dots X_d \}$ and $\mathcal{Y} = \{ Y_1 \dots Y_m \}$

discrete random variables, $X_i \in [n_i]$ and $Y_i \in [r_i]$

such that $\mathcal{X}$ is independent conditionally to $\mathcal{Y}$

and viceversa.

$$\mathbb{P}(\mathcal{X} = \underline{i}) = \sum_{\underline{j} \in [\underline{r}]} \mathbb{P}(\mathcal{Y} = \underline{j})\, \mathbb{P}(\mathcal{X} = \underline{i} \mid \mathcal{Y} = \underline{j}) =$$

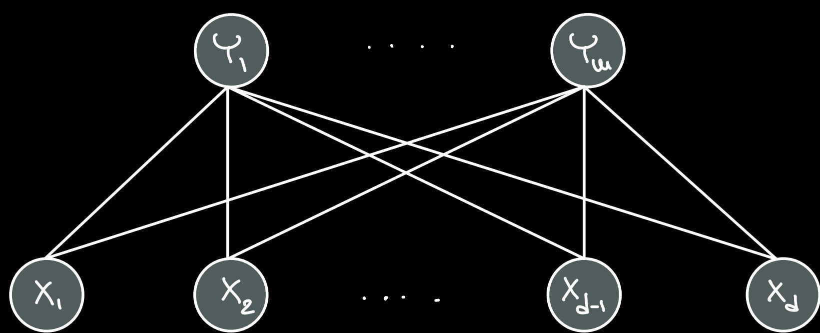
$$= \sum_{\underline{j} \in [\underline{r}]} \mathbb{P}(\mathcal{Y} = \underline{j})\, \mathbb{P}(X_1 = i_1 \mid \mathcal{Y} = \underline{j}) \cdots \mathbb{P}(X_d = i_d \mid \mathcal{Y} = \underline{j})$$

$$= \prod_{h \in [m]} \sum_{j_h \in [r_h]} \mathbb{P}(Y_h = j_h)\, \mathbb{P}(X_1 = i_1 \mid Y_h = j_h) \cdots \mathbb{P}(X_d = i_d \mid Y_h = j_h)$$

PRODUCTS OF MIXTURES OF INDEPENDENCE MODELS
(aka RESTRICTED BOLTZMANN MACHINES)

Let $\mathcal{X} = \{ X_1 \dots X_d \}$ and $\mathcal{Y} = \{ Y_1 \dots Y_m \}$ discrete random variables, $X_i \in [n_i]$ and $Y_i \in [r_i]$ such that $\mathcal{X}$ is independent conditionally to $\mathcal{Y}$ and viceversa.

$$T_{\mathcal{X}|\mathcal{Y}} = T_{\mathcal{X}|Y_1} * \cdots * T_{\mathcal{X}|Y_m}$$



hidden layer

visible layer

"Restricted Boltzmann Machines: Introduction and Review" G. Montúfar (2016)

HADAMARD PRODUCTS OF SECANT VARIETIES

Hadamard products of varieties

$x = (x_0 : \dots : x_n)$, $y = (y_0 : \dots : y_n) \in \mathbb{P}^n$

If exists,

$$x * y = (x_0 y_0 : \dots : x_n y_n)$$

$X, Y \subseteq \mathbb{P}V$

$$X * Y = \overline{\{ x * y \ : \ x \in X, \ y \in Y, \ x * y \text{ exists} \}}$$

"Geometry of Restricted Boltzmann Machines", M.A. Cueto, J. Morton, B. Sturmfels (2010)

HADAMARD PRODUCTS OF SECANT VARIETIES

Hadamard products of varieties

$x = (x_0 : \dots : x_n)$, $y = (y_0 : \dots : y_n) \in \mathbb{P}^n$

If exists ,

$$x * y = (x_0 y_0 : \dots : x_n y_n)$$

$X, Y \subseteq \mathbb{P}V$

$$X * Y = \overline{\{ x * y \ : \ x \in X , y \in Y , x * y \text{ exists} \}}$$

⚠ This definition is highly dependent on the variables ⚠

"Geometry of Restricted Boltzmann Machines", M.A. Cueto, J. Morton, B. Sturmfels (2010)

HADAMARD PRODUCTS OF SECANT VARIETIES

but we look at tensors as multidimensional arrays encoding some fixed data, so in a fixed embedding

Secant varieties of Segre varieties

$$\sigma_r \, \mathrm{Seg}(\underline{n}) = \overline{\left\{ T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d} \; : \; \mathrm{rank}(T) \le r \right\}}$$

Hadamard products of secant varieties

$$\sigma_{\underline{r}} \, \mathrm{Seg}(\underline{n}) = \overline{\left\{ T = T_1 * \cdots * T_m \; : \; \mathrm{rank}(T_i) \le r_i \right\}}$$

$$= \sigma_{r_1} \mathrm{Seg}(\underline{n}) * \cdots * \sigma_{r_m} \mathrm{Seg}(\underline{n})$$

"Geometry of Restricted Boltzmann Machines", M.A. Cueto, J. Morton, B. Sturmfels (2010)

AN IDENTIFIABILITY RESULT

Q

Assume we have a (generic) tensor $T \in \sigma_{\underline{r}} \, Seg(\underline{n})$,

i.e.,
$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{i,j} \qquad \mathrm{rank}(A_{ij}) = 1 ,$$

can we recover the A_{ij}'s from T?

AN IDENTIFIABILITY RESULT

Q

Assume we have a (generic) tensor $T \in \sigma_{\underline{r}}\, Seg(\underline{n})$,

i.e.,
$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} \qquad \mathrm{rank}(A_{ij}) = 1,$$

can we recover the A_{ij}'s from T?

Note that :

(1) Hadamard product of rank-one tensors is of rank-one

$$(a_1 \otimes \cdots \otimes a_d) * (b_1 \otimes \cdots \otimes b_d) = (a_1 * b_1) \otimes \cdots \otimes (a_d * b_d)$$

More in general, if X is toric variety, then $X * X = X$.

An Identifiability Result

Q

Assume we have a (generic) tensor $T \in \sigma_{\underline{r}}\, Seg(\underline{n})$,

i.e.,
$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{i,j} \quad , \quad rank(A_{ij}) = 1 ,$$

can we recover the A_{ij}'s from T?

Note that:

(1) Hadamard product of rank-one tensors is of rank-one

$$(a_1 \otimes \cdots \otimes a_d) \ast (b_1 \otimes \cdots \otimes b_d) = (a_1 \ast b_1) \otimes \cdots \otimes (a_d \ast b_d)$$

More in general, if X is toric variety, then $X \ast X = X$.

Hence, if $T = T_1 \ast \cdots \ast T_m$ is a $\underline{r}$-HHD,

then $T = T_1 \ast \cdots \ast (A \ast T_i) \ast \cdots \ast (A^{\ast(-1)} \ast T_j) \ast \cdots \ast T_m$

is a $\underline{r}$-HHD $\forall A$ of rank-one and with non-zero entries

An Identifiability Result

Q

Assume we have a (generic) tensor $T \in \sigma_{\underline{r}} \, \mathrm{Seg}(\underline{n})$,

i.e.,
$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} \quad , \quad \mathrm{rank}(A_{ij}) = 1 \, ,$$

can we recover the A_{ij}'s from T?

Note that:

(1) Hadamard product of rank-one tensors is of rank-one

$$\text{exp.dim } \sigma_{\underline{r}}(S) = \min \left\{ \dim \mathbb{P}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}) \, , \; \sum_{i=1}^{m} \dim \sigma_{r_i}(S) - (m-1) \dim S - 1 \right\}$$

and this is always an upper bound.

"Hadamard products of linear spaces" C. Bocci, E. Carlini, J. Kileel (2017)

An Identifiability Result

Q

Assume we have a (generic) tensor $T \in \sigma_{\underline{r}}\, Seg(\underline{u})$,

i.e.,
$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{i,j} \quad , \quad \text{rank}(A_{ij}) = 1 \,,$$

can we recover the A_{ij}'s from T?

Note that:

(2) a $\underline{r}$-HHD induces a Hitchcock decomposition

By distributivity,

$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \sum_{\underline{j} \in [\underline{r}]} \left(A_{1,j_1} \ast \cdots \ast A_{m,j_m} \right)$$

Identifiability of Hitchcock decompositions

Let $S = \mathrm{Seg}(\underline{n})$.

$S^{(R)} := S^{\times R} / \mathfrak{S}_R$

$$\varphi_R : S^{(R)} \times \mathbb{P}\mathbb{C}^R \longrightarrow \mathbb{P}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d})$$

$$(\{[A_1] \cdots [A_R]\}, (\lambda_1 : \cdots : \lambda_R)) \mapsto [\lambda_1 A_1 + \cdots + \lambda_R A_R]$$

Decomposition locus of $T \in \mathrm{im}(\varphi_R)$:

$$D_R(T) := \pi_1(\varphi_R^{-1}([T]))$$

T is R-identifiable if $D_R(T)$ is a singleton

"Three-way arrays: rank and uniqueness of trilinear decompositions" J. Kruskal (1977)

"Effective criteria for specific identifiability of tensors and forms" L. Chiantini, G. Ottaviani, N. Vannieuwenhoven (2017)

"A generalization of Kruskal's Theorem on tensor decomposition" B. Lovitz, F. Petrov (2021)

Identifiability of Hadamard-Hitchcock decompositions

Let $S = \mathrm{Seg}(\underline{n})$ and $\underline{r} = (r_1 \dots r_m)$

$\mathfrak{S}(\underline{r}) \subseteq \mathfrak{S}_m$: permutations preserving $\underline{r}$

$\mathfrak{S}(\underline{r}) = \mathfrak{S}^{(r_1)} \times \dots \times \mathfrak{S}^{(r_m)} / \mathfrak{S}(\underline{r})$

Given a $\underline{r}$-HHD, $T = \mathop{\ast}\limits_{j=1}^{m} \sum_{i=1}^{r_j} A_{i,j}$, we can:

- permute summands in each Hadamard factor

Identifiability of Hadamard-Hitchcock decompositions

Let $S = \mathrm{Seg}(\underline{n})$ and $\underline{r} = (r_1 \dots r_m)$

$\mathfrak{S}(\underline{r}) \subseteq \mathfrak{S}_m$: permutations preserving $\underline{r}$

$S(\underline{r}) = S^{(r_1)} \times \dots \times S^{(r_m)} / \mathfrak{S}(\underline{r})$

Given a $\underline{r}$-HHD, $T = \mathop{\ast}\limits_{j=1}^{m} \sum_{i=1}^{r_j} A_{i,j}$, we can:

- permute summands in each Hadamard factor
- permute Hadamard factors accordingly to $\mathfrak{S}(\underline{r})$

Identifiability of Hadamard-Hitchcock decompositions

Let $S = \mathrm{Seg}(\underline{v})$ and $\underline{r} = (r_1 \dots r_m)$

$\mathfrak{S}(\underline{r}) \subseteq \mathfrak{S}_m$: permutations preserving $\underline{r}$

$S^{(\underline{r})} = S^{(r_1)} \times \dots \times S^{(r_m)} / \mathfrak{S}(\underline{r})$

Given a $\underline{r}$-HHD , $T = \mathop{\ast}\limits_{j=1}^{m} \sum_{i=1}^{r_j} A_{i,j}$, we can :

- permute summands in each Hadamard factor
- permute Hadamard factors accordingly to $\mathfrak{S}(\underline{r})$

$$\mathcal{H}_{\underline{r}}(T) = \left\{ [(A_{ij})_{ij}] \in S^{(\underline{r})} \ : \ T = \mathop{\ast}\limits_{j=1}^{m} \sum_{i=1}^{r_j} A_{ij} \right\}$$

... but this is not enough

Identifiability of Hadamard-Hitchcock decompositions

Let $S^* = \{ T \in S : T$ has non-zero entries $\}$.

This is a group with respect to the Hadamard product.

- If $A \in S^*$ and $[(A_{ij})_{ij}] \in \mathcal{H}_{\underline{r}}(T)$, then $\forall h \neq k$

$$[(\cdots(A * A_{h,j})_j \cdots (A^{*(-1)} * A_{k,j})_j \cdots)] \in \mathcal{H}_{\underline{r}}(T)$$

Identifiability of Hadamard-Hitchcock decompositions

Let $S^* = \{ T \in S : T$ has no-zero entries $\}$.

This is a group with respect to the Hadamard product.

- If $A \in S^*$ and $[(A_{ij})_{ij}] \in \mathcal{H}_{\underline{r}}(T)$, then $\forall h \neq k$

$$[(\cdots(A * A_{h,j})_j \cdots (A^{*(-1)} * A_{k,j})_j \cdots)] \in \mathcal{H}_{\underline{r}}(T)$$

Hadamard-Hitchcock decomposition locus of T is

$$\mathcal{H}_{\underline{r}}(T)^{S^*} = \mathcal{H}_{\underline{r}}(T) / S^*$$

T is said to be $\underline{r}$-identifiable if $\mathcal{H}_{\underline{r}}(T)^{S^*}$ is singleton.

Theorem (Oneto-Vannieuwenhoven)

Let $\underline{r} = (r_1 \cdots r_m)$ and $R = r_1 \cdots r_m$.

Let T be a generic tensor of rank-R which is identifiable.

If $\mathcal{H}_{\underline{r}}(T) \neq \emptyset$, then T is $\underline{r}$-identifiable.

"Effective criteria for specific identifiability of tensors and forms" L. Chiantini, G. Ottaviani, N. Vannieuwenhoven (2017)

Theorem (Oneto-Vannieuwenhoven)

Let $\underline{r} = (r_1 \cdots r_m)$ and $R = r_1 \cdots r_m$.

Let T be a generic tensor of rank-R which is identifiable.

If $\mathcal{H}_{\underline{r}}(T) \neq \phi$, then T is $\underline{r}$-identifiable.

Corollary

Let $\underline{r} = (r_1 \cdots r_m)$ and $R = r_1 \cdots r_m$.

Let $\underline{n} = (n_1 \cdots n_d)$ and, for $X \in [d]$, $N_X = \prod_{x \in X} n_x$.

Assume that there is a partition $I \sqcup J \sqcup K = [d]$ s.t.

$$N_I \geq N_J \geq N_K \geq 2 \quad \text{and} \quad R \leq N_I + \min\left\{\tfrac{1}{2}\delta, \delta\right\}$$

$$\delta = N_K + N_J - N_I - 2$$

then generic $\underline{r}$-identifiability holds in $\sigma_{\underline{r}}(S)$

"Effective criteria for specific identifiability of tensors and forms" L. Chiantini, G. Ottaviani, N. Vannieuwenhoven (2017)

Theorem (Oneto-Vannieuwenhoven)

Let $\underline{r} = (r_1 \cdots r_m)$ and $R = r_1 \cdots r_m$.

Let T be a generic tensor of rank-R which is identifiable.

If $\mathcal{H}_{\underline{r}}(T) \neq \phi$, then T is $\underline{r}$-identifiable.

Corollary

Let $\underline{r} = (r_1 \cdots r_m)$ and $R = r_1 \cdots r_m$.

Let $\underline{n} = (n_1 \cdots n_d)$ and, for $X \in [d]$, $N_X = \prod_{x \in X} n_x$.

Assume that there is a partition $I \sqcup J \sqcup K = [d]$ s.t.

$$N_I \geq N_J \geq N_K \geq 2 \quad \text{and} \quad R \leq N_I + \min\left\{\frac{1}{2}\delta, \delta\right\}$$

$$\delta = N_K + N_J - N_I - 2$$

then generic $\underline{r}$-identifiability holds in $\sigma_{\underline{r}}(S)$

and

$$\dim \sigma_{\underline{r}}(S) = \sum_{i=1}^{m} \dim \sigma_{r_i}(S) - (m-1)\dim(S)$$

Idea of proof

Assume
$$T = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i'} B_{ij} .$$

Then, we induce

$$T = \sum_{\underline{j} \in [\underline{r}]} A_{1j_1} * \cdots * A_{m,j_m} = \sum_{\underline{j} \in [\underline{r}]} B_{1j_1} * \cdots * B_{m,j_m}$$

Idea of proof

Assume

$$T = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i'} B_{ij} .$$

Then, we induce

$$T = \sum_{\underline{j} \in [r]} A_{1j_1} * \cdots * A_{m,j_m} = \sum_{\underline{j} \in [r]} B_{1j_1} * \cdots * B_{m,j_m}$$

By R-identifiability, $\exists\, \pi \in \mathfrak{S}([r])$, $(k_1 \ldots k_m) = \pi(j_1 \ldots j_m)$

$$A_{1j_1} * \cdots * A_{m,j_m} = B_{1,k_1} * \cdots * B_{m,k_m}$$

Idea of proof

Assume

$$T = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \mathop{*}_{i=1}^{m} \sum_{j=1}^{r_i'} B_{ij} .$$

Then, we induce

$$T = \sum_{\underline{j} \in [\underline{r}]} A_{1j_1} * \cdots * A_{m,j_m} = \sum_{\underline{j} \in [\underline{r}]} B_{1j_1} * \cdots * B_{m,j_m}$$

By R-identifiability, $\exists \pi \in \mathfrak{S}([\underline{r}])$, $(k_1 \dots k_m) = \pi(j_1 \cdots j_m)$

$$A_{1j_1} * \cdots * A_{m,j_m} = B_{1,k_1} * \cdots * B_{m,k_m}$$

We show :

- π is induced by an element in $\mathfrak{S}([\underline{r}]) \times \mathfrak{S}(r_1) \times \cdots \times \mathfrak{S}(r_m)$
- For every $i \in [m]$, $\left(\sum_{j=1}^{r_i} A_{ij} \right) * \left(\sum_{j=1}^{r_i} B_{ij} \right)^{*(-1)}$ has rank one

We restrict to only one variable to study the permutation π.

Say $a_{ij} = [A_{ij}]_{(1\dots 1)}$, $b_{ij} = [B_{ij}]_{(1\dots 1)}$.

Then, we get

$$(\dots, a_{1j_1} \cdots a_{mj_m}, \dots) = (\dots, b_{1k_1} \cdots b_{mk_m}, \dots) \in \mathbb{C}^{r_1} \otimes \cdots \otimes \mathbb{C}^{r_m}.$$

By genericity : π is a "rank-1 preserver"

i.e., π maps generic rank-1 tensors to rank-1

[Westwick, 1967]

$\pi = (f_1 \otimes \cdots \otimes f_m) \circ \rho$ where $f_i \in \mathbb{C}^{r_i} \otimes (\mathbb{C}^*)^{r_i}$

$\rho \in \mathfrak{S}(\underline{r})$

"Transformation of tensor spaces" R. Westwick (1967)

In particular, we show $f_i = d_i^{(1\dots1)}\,\pi_i \quad \forall i \in [m]$

where $\pi_i \in \mathfrak{S}(r_i)$ with $d_1^{(1\dots1)} \cdots d_m^{(1\dots1)} = 1$

for a different entry $\underline{i} = (i_1 \dots i_d) \in [\underline{n}]$

the permutation π_i would be the same,

but we might have a different $f_i = d_i^{\underline{i}} \cdot \pi_i$.

In particular, we show $f_i = d_i^{(1\dots1)} \pi_i \quad \forall i \in [m]$

where $\pi_i \in \mathfrak{S}(r_i)$ with $d_1^{(1\dots1)} \cdots d_m^{(1\dots1)} = 1$

for a different entry $\underline{i} = (i_1 \dots i_d) \in [\underline{n}]$
the permutation π_i would be the same,
but we might have a different $f_i = d_i^{\underline{i}} \cdot \pi_i$.

Hence,

$$\big[\dots A_{ij} \dots\big] = \big[\dots, \underset{\substack{\downarrow \\ \text{depend only on } i \in [m]}}{D_i} * B_{p(i),\pi_i(j)}, \dots\big] \in \mathcal{H}_{\underline{r}}(T)^{S^*}$$

where $D_i = \left(d_i^{(i_1 \dots i_d)}\right) \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$

$$\left(\sum_{j=1}^{r_i} A_{ij}\right) * \left(\sum_{j=1}^{r_i} B_{ij}\right)^{*(-1)} = D_i = A_{ij} * B_{p(i),\pi_i(j)}^{*(-1)} \text{ has rank-one}$$

#

FROM HITCHCOCK TO HADAMARD-HITCHCOCK

Our approach suggests the following algorithm to compute a r-HHD of a generic tensor.

FROM HITCHCOCK TO HADAMARD-HITCHCOCK

Our approach suggests the following algorithm to compute a $\underline{r}$-HHD of a generic tensor.

Given an induced Hitchcock decomposition

$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \sum_{\underline{j}} A_{1j_1} \ast \cdots \ast A_{mj_m} = \sum_{\underline{j} \in [\underline{r}]} A_{\underline{j}}$$

consider the tensor

$$[H]_{\underline{i},\underline{j}} = [A_{1j_1} \ast \cdots \ast A_{mj_m}]_{\underline{i}} \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d} \otimes \mathbb{C}^{r_1} \otimes \cdots \otimes \mathbb{C}^{r_m}$$

FROM HITCHCOCK TO HADAMARD-HITCHCOCK

Our approach suggests the following algorithm
to compute a $\underline{r}$-HHD of a generic tensor.

Given an induced Hitchcock decomposition

$$T = \mathop{\ast}_{i=1}^{m} \sum_{j=1}^{r_i} A_{ij} = \sum_{\underline{j}} A_{1j_1} \ast \cdots \ast A_{mj_m} = \sum_{\underline{j} \in [\underline{r}]} A_{\underline{j}}$$

consider the tensor

$$[H]_{\underline{i},\underline{j}} = [A_{1j_1} \ast \cdots \ast A_{mj_m}]_{\underline{i}} \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d} \otimes \mathbb{C}^{r_1} \otimes \cdots \otimes \mathbb{C}^{r_m}$$

- $[H]_{\bullet,\underline{j}} = A_{1j_1} \ast \cdots \ast A_{mj_m} \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$
- $[H]_{\underline{i},\bullet} = t_{\underline{i}}^{1} \otimes \cdots \otimes t_{\underline{i}}^{m} \in \mathbb{C}^{r_1} \otimes \cdots \otimes \mathbb{C}^{r_m}$

both of rank one

where $t_{\underline{i}}^{\ell} = ((A_{\ell,1})_{\underline{i}}, \ldots, (A_{\ell,r_\ell})_{\underline{i}}) \in \mathbb{C}^{r_\ell}$

From Hitchcock to Hadamard-Hitchcock

Our approach suggests the following algorithm to compute a $\underline{r}$-HHD of a generic tensor.

Given an induced Hitchcock decomposition

$$T = \qquad\qquad = \sum_{\underline{j}\in[\underline{r}]} A_{\underline{j}}$$

consider the tensor

$$[H]_{\underline{i},\underline{j}} = [\ A_{\underline{j}}\ ]_{\underline{i}} \in \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_d}\otimes\mathbb{C}^{r_1}\otimes\cdots\otimes\mathbb{C}^{r_m}$$

- $[H]_{\bullet,\underline{j}} = A_{\underline{j}} \in \mathbb{C}^{n_1}\otimes\cdots\otimes\mathbb{C}^{n_d}$ $\longleftarrow$ rank-one
- $[H]_{\underline{i},\bullet} = ???\ \in \mathbb{C}^{r_1}\otimes\cdots\otimes\mathbb{C}^{r_m}$ $\longleftarrow$ rank-one?

FROM HITCHCOCK TO HADAMARD-HITCHCOCK

Our approach suggests the following algorithm to compute a $\underline{r}$-HHD of a generic tensor.

1. Find a rank-R Hitchcock decomposition $T = A_1 + \cdots + A_R$ and certify its uniqueness

2. Find a permutation such that
$$H = \left(A_{\pi(\underline{i}),\underline{j}}\right) \in \mathbb{C}^{(n_1 \times \cdots \times n_d) \times (r_1 \times \cdots \times r_d)}$$
has $(\cdot, \underline{j})$ and $(\underline{i}, \cdot)$ slices of rank-one

3. Find rank-one decomposition of each row $H_{\underline{i},*}$ to deduce the $\underline{r}$-HHD.

RANK-1 PERMUTATIONS (SUDOKU FOR EXPERTS)

It is equivalent to the following question

Q

Let $\underline{r} = (r_1 \dots r_m)$ and $R = r_1 \cdots r_m$.

Given $\underline{a} = (a_1 \dots a_R)$, is there a map $\pi : [R] \to [\underline{r}]$ such that the induced $\pi(\underline{a}) \in \mathbb{C}^{r_1} \otimes \dots \otimes \mathbb{C}^{r_m}$ is rank-one?

Without loss of generality we may assume

- $a_1 \geq a_2 \geq \dots \geq a_R$
- $\pi(1) = (1, \dots, 1)$

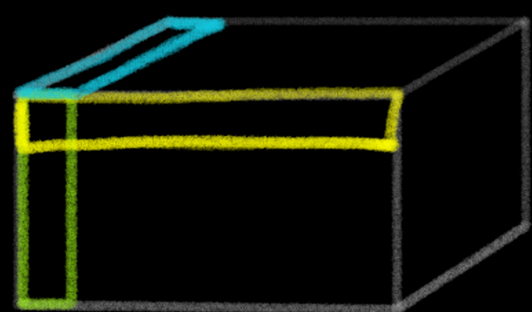
RANK-1 PERMUTATIONS

Note that if $\operatorname{rank}(A) = 1$, then

$$A = a_1 \begin{bmatrix} 1 \\ A_{21\dots1}\cdot a_1^{-1} \\ \vdots \\ A_{r_1 1\dots 1}\cdot a_1^{-1}\end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ A_{1\dots 2}\cdot a_1^{-1} \\ \vdots \\ A_{1\dots r_m}\cdot a_1^{-1}\end{bmatrix}$$

Then, it is enough to determine the cross of A:

$$C_{1\dots1}(A) = \underbrace{\{A_{i_1 1\dots1} : i_1\in[r_1]\}}_{\shortparallel \atop C^1_{1\dots1}(A)} \cup\dots\cup \underbrace{\{A_{1\dots1 i_m} : i_m\in[r_m]\}}_{\shortparallel \atop C^m_{1\dots1}(A)}$$



RANK-1 PERMUTATIONS

Recall that

$T \in \mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_d}$ has rank 1

if and only if

all flattenings $T_j : (\mathbb{C}^{n_j})^* \longrightarrow \mathbb{C}^{n_1} \otimes \cdots \widehat{\mathbb{C}^{n_j}} \cdots \otimes \mathbb{C}^{n_d}$

have rank 1

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1 \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

$$\mathcal{M}(\underline{a}) = \left\{ m_{ijk} = \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} : \det m_{ijk} = 0 \right\}$$

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1 \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

$$\mathcal{M}(\underline{a}) = \left\{ m_{ijk} = \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} : \det m_{ijk} = 0 \right\}$$

Lemma 1

$$\{ a_i : i \in [R] \} \setminus C_{1\dots1}(A) = \{ a_k : \exists\, ij \in [R] \ \det m_{ijk} = 0 \}$$

the elements of the $\underline{1}$-cross have Hamming distance one from position $\underline{1}$ and flattening can only decrease the Hamming distance

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1 \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

$$\mathcal{M}(\underline{a}) = \left\{ m_{ijk} = \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} : \det m_{ijk} = 0 \right\}$$

Lemma 1

$$\{a_i : i \in [R]\} \setminus C_{1\dots1}(A) = \{a_k : \exists\, i j \in [R] \ \det m_{ijk} = 0\}$$

Lemma 2

The h-th direction of cross is first column of h-th flattening;

i.e., $\{a_i, a_j\} \subseteq C^h_{1\dots1}(A)$ if and only if

$\forall k \neq i,j$, $\det m_{ijk} \neq 0$.

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1, \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

(i) Compute the pairs $(i,j) \in [(R,R)]$ $i \neq j$ s.t.

$$a_i a_j a_1^{-1} = a_k \text{ for some } k.$$

$$\Rightarrow \quad \det \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} = 0 \quad \text{and} \quad a_k \notin C_{1\dots\ell}(A)$$

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1 \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

(1) Compute the pairs $(i,j) \in [(R,R)]$ $i \neq j$ s.t.

$$a_i a_j a_1^{-1} = a_k \text{ for some } k.$$

$$\Rightarrow \quad \det \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} = 0 \quad \text{and} \quad a_k \notin C_{1\dots\ell}(A)$$

(2) Fixed $a_i \in C_{1\dots\ell}(A)$,

we filter out all elements a_j s.t.

$$a_i a_j a_i^{-1} = a_k \text{ for some } k$$

$\Rightarrow$ a_j belongs to the same direction of a_i

RANK-1 PERMUTATIONS

Let $\underline{a} = (a_1 \dots a_R)$ generic and $A = \pi(\underline{a})$ of rank-one.

(1) Compute the pairs $(i,j) \in [(R,R)]$ $i \neq j$ s.t.

$$a_i a_j a_1^{-1} = a_k \text{ for some } k.$$

$$\Rightarrow \quad \det \begin{bmatrix} a_1 & a_j \\ a_i & a_k \end{bmatrix} = 0 \quad \text{and} \quad a_k \notin C_{1\dots\ell}(A)$$

(2) Fixed $a_i \in C_{1\dots\ell}(A)$,

we filter out all elements a_j s.t.

$$a_i a_j a_1^{-1} = a_k \text{ for some } k.$$

$\Rightarrow$ a_j belongs to the same direction of a_i

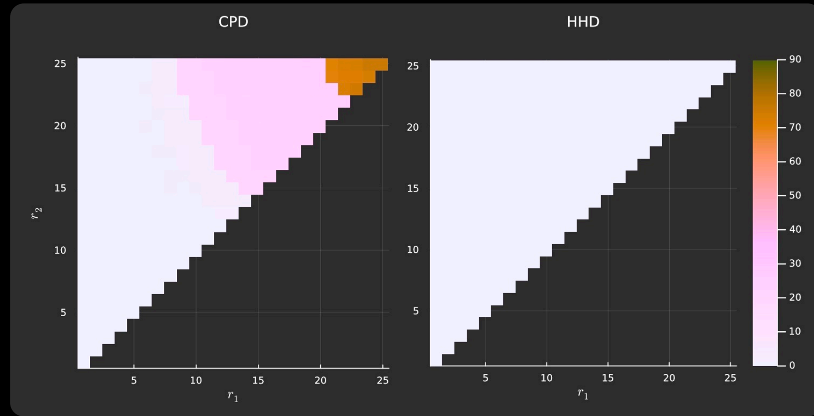
(3) We proceed with (2) until we determine all directions

EXPERIMENTS

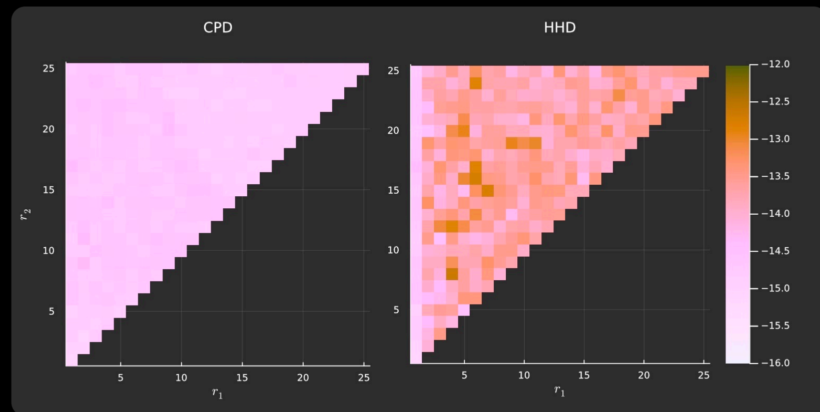
```
Performing CPD decomposition:
0. Grouping [[2, 4, 5], [1], [3]] and reshaped to (47250, 50, 40) tensor
    in 0.318342279 s.
1. Performed ST-HOSVD compression to size (945, 50, 40) in 9.316073541 s.
  Swapped factors 2 and 3, so the tensor has size (945, 40, 50)
  Selected degree increment d_0 = [1, 0]
2. Constructed kernel of A_1 of size (945, 2000) in 5.12821185 s.
3. Constructed resultant map of size (41000, 42200) in 96.356968786 s.
4. Constructed res res' in 184.738217429 s.
5. Computed cokernel of size (945, 41000) in 1235.68512347 s.
6. Constructed multiplication matrices in 1.783130807 s.
7. Diagonalized multiplication matrices and extracted solution in 6.991025653 s.
8. Refined factor matrices Y and Z in 49.639239108 s.
9. Recovered factor matrix X in 0.114635626 s.
10. Recovered the full factor matrices in 2.282236235 s.
>> Computed a rank-945 CPD of a (50, 45, 40, 35, 30) tensor in 1592.642372846s.

Performing CPD to HHD decomposition:
1. Computed fixed entry of all rank-1 tensors in 1.6719e-5s.
2. Found all tensor 1-cross 2x2 minors in 0.149103568s.
  Rank-1 permutation reconstruction relative backward error = 2.6547305725693787e-15
3. Reconstructed rank-1 permutation in 0.000182103s.
4. Decoupled 945 rank-1 tensors into [3, 5, 7, 9] rank-1 tensors in 0.002535851s.
>> Decomposed rank-945 CPD into a rank-[3, 5, 7, 9] HHD in 0.151891268s.

Information:
- CPD relative backward error: 1.7698167026458318e-15
- HHD relative backward error: 2.745738593311826e-13
- Extra lop: 2.190730898636927
- Total computation time: 2058.179259416s.
```



Computational Time



Backward Error

Experiment with Julia

$\underline{n} = (50, 45, 40, 35, 30)$

$\underline{r} = (3, 5, 7, 9)$

Thank you !