Identifiability of centered Gaussian Mixtures and sums of powers of quadratics

Alessandro Oneto

with A. Taveira Blomenhofer, A. Casarotti, M. Michałek

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Pearson's crabs: algebraic statistics in 1893

(1983)
K. Pearson wanted to explain an asymmetry observed in data about crabs in Naples'

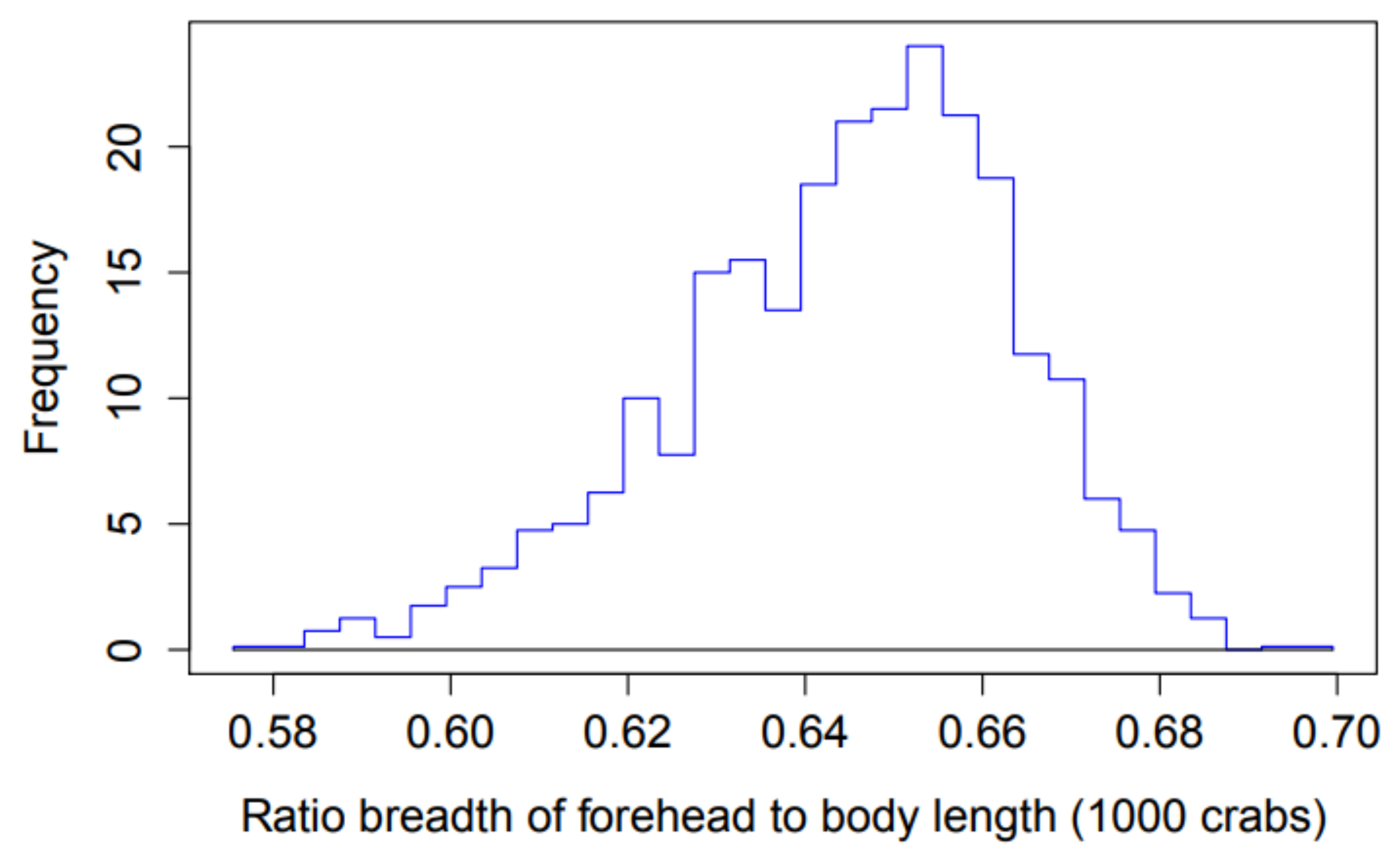

Figure 1.1.1.: Histogram for Crabs Data

III. *Contributions to the Mathematical Theory of Evolution.*

By KARL PEARSON, *University College, London.*

Communicated by Professor HENRICI, *F.R.S.*

Received October 18,—Read November 16, 1893.

normal shape, and it becomes important to determine the direction and amount of such deviation. The asymmetry may arise from the fact that the units grouped together in the measured material are not really homogeneous. It may happen that we have a mixture of 2, 3, . . . n homogeneous groups, each of which deviates about its own mean symmetrically and in a manner represented with sufficient accuracy by the normal curve. Thus an abnormal frequency-curve may be really built up of normal curves having parallel but not necessarily coincident axes and different parameters.

K. Pearson. *Contribution to the Mathematical Theory of Evolution.* Philosophical Transactions of the Royal Society. 1893
C. Amendóla. *Algebraic Statistics of Gaussian Mixtures.* PhD thesis, TU Berlin. 2017

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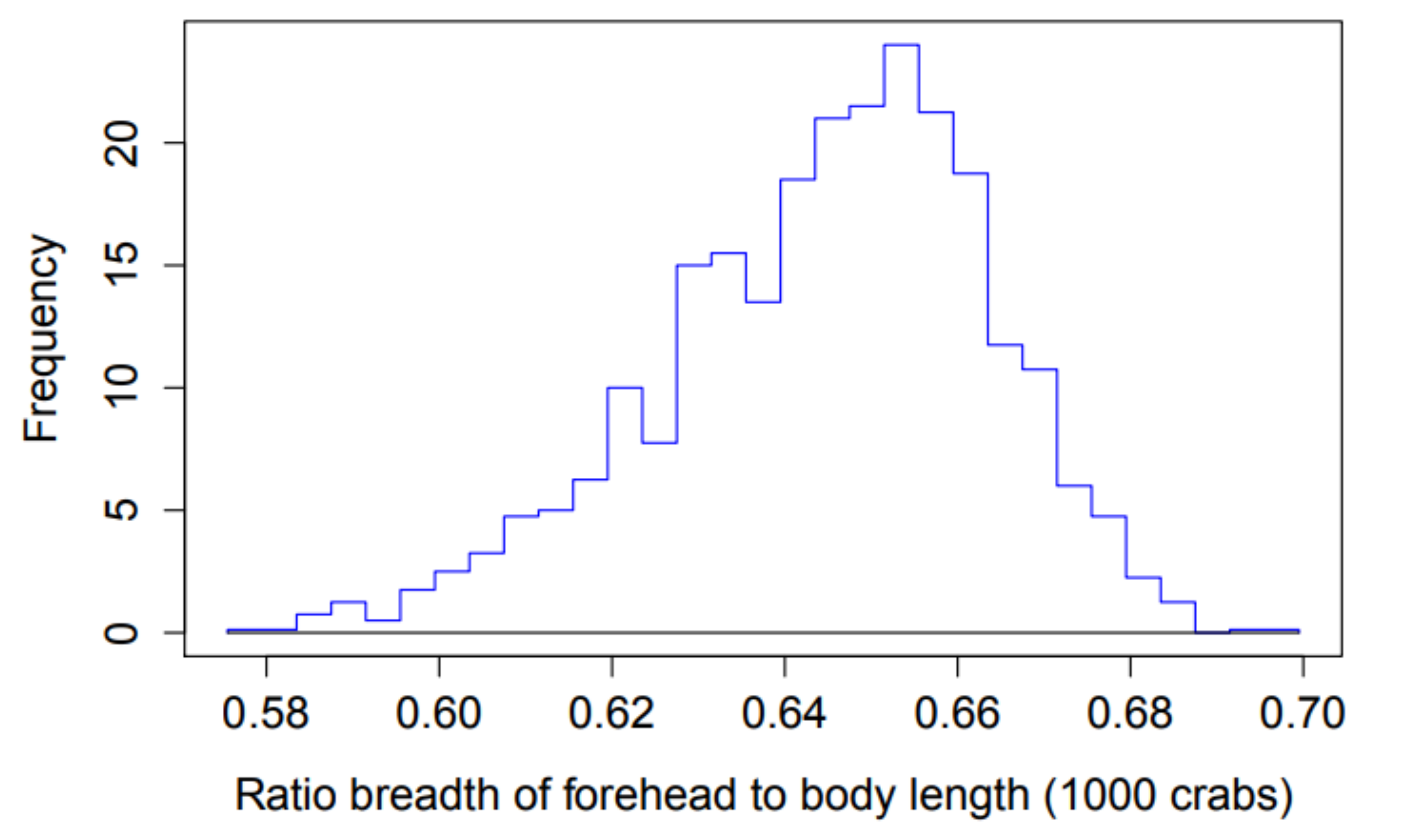

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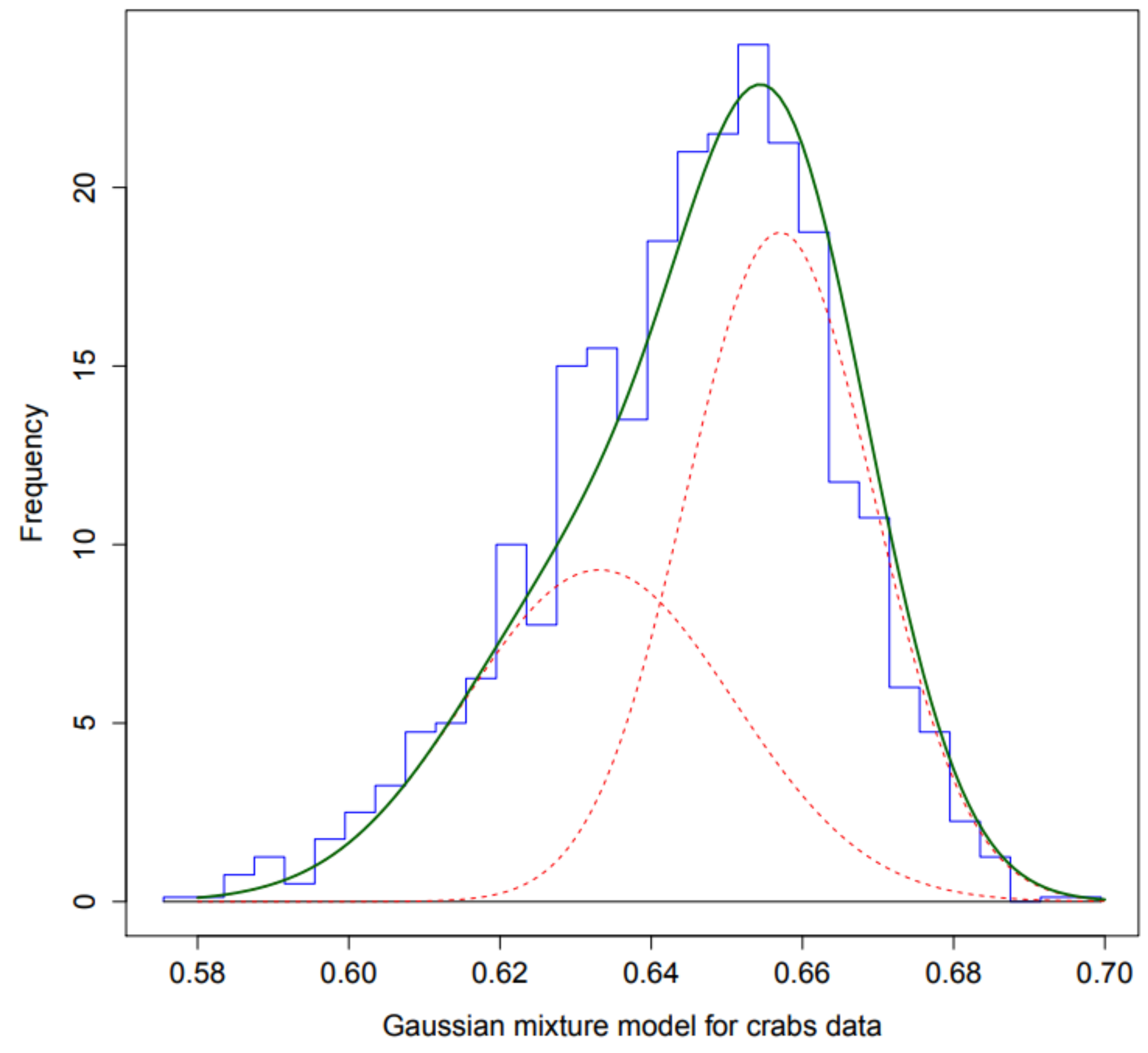

Figure 4.2.1.: Approximation of crabs data by a mixture of two Gaussians.

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Method of moments.

$$z_1 + z_2 = 1 \quad \ldots \quad (8).$$

$$\gamma_1 z_1 + \gamma_2 z_2 = 0 \quad \ldots \quad (9).$$

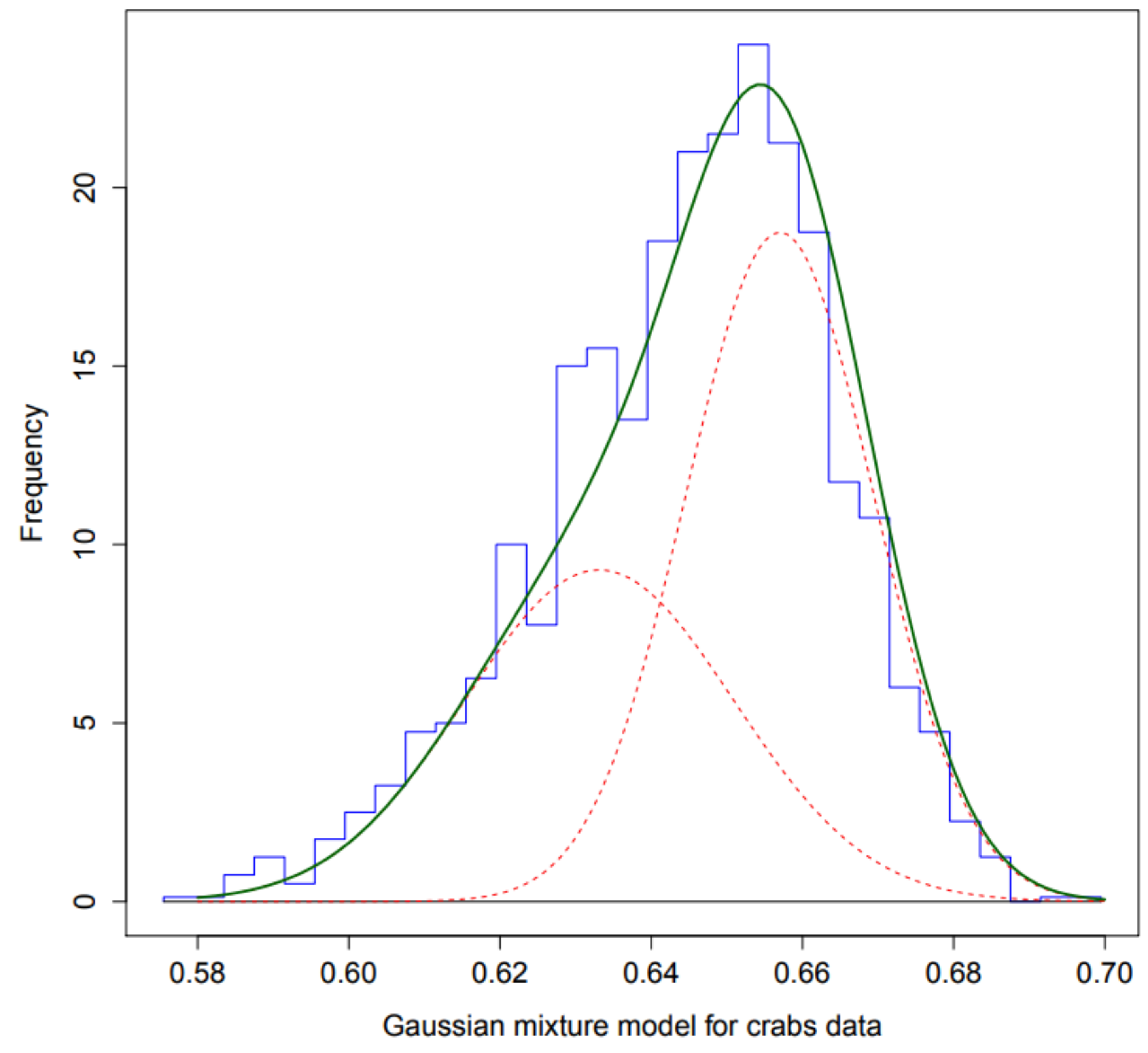
$$\gamma_1^2 z_1 (1 + u_1^2) + \gamma_2^2 z_2 (1 + u_2^2) = \mu_2 \quad \ldots \quad (10).$$

$$\gamma_1^3 z_1 (1 + 3u_1^2) + \gamma_2^3 z_2 (1 + 3u_2^2) = \mu_3 \quad \ldots \quad (11).$$

$$\gamma_1^4 z_1 (1 + 6u_1^2 + 3u_1^4) + \gamma_2^4 z_2 (1 + 6u_2^2 + 3u_2^4) = \mu_4 \quad \ldots \quad (12).$$

$$\gamma_1^5 z_1 (1 + 10u_1^2 + 15u_1^4) + \gamma_2^5 z_2 (1 + 10u_2^2 + 15u_2^4) = \mu_5 \quad \ldots \quad (13).$$

Equations (8)–(13) give the complete solution of the problem.* After several trials, I find that the elimination of z_1, z_2, u_1, u_2 from these equations, and the determination of equations giving $\gamma_1\gamma_2$ and $\gamma_1 + \gamma_2$ appear to lead to a resulting equation of the lowest possible order.



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Moments

Consider a random variable $X = (X_1, \ldots, X_n)$ defined by a density function $f(X) : \mathbb{R}^n \to \mathbb{R}$.

For $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$, the **i-moment** of X is:

$$m_{\mathbf{i}} = \mathbb{E}[X_1^{i_1} \cdots X_n^{i_n}] = \int_{\mathbb{R}^n} x^{\mathbf{i}} f(x) dx.$$

For $d \in \mathbb{N}$, the **d-moment** of X is the degree-d polynomial:

$$m_d = \mathbb{E}[\langle X, Y \rangle^d] = \sum_{\mathbf{i} \in \mathbb{N}^n,\ i_1 + \ldots + i_n = d} \frac{d!}{i_1! \cdots i_n!} m_{\mathbf{i}} Y^{\mathbf{i}} \in \mathbb{R}[Y].$$

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A compact way to look at moments is via the **moment generating function:**

$$\mathbb{E}[\exp(\langle X, Y \rangle)] = \sum_d \frac{1}{d!} \mathbb{E}[\langle X, Y \rangle^d] = \sum_d \frac{1}{d!} m_d \in \mathbb{R}[[Y]].$$

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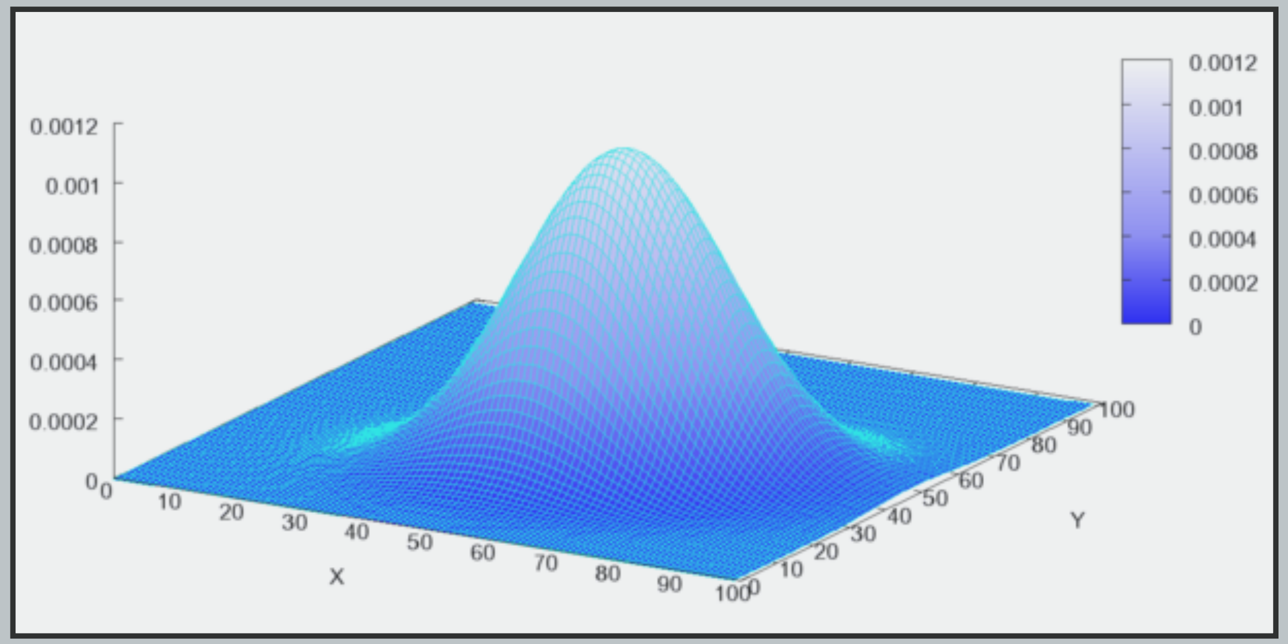
Moments of Gaussian distributions

Consider a multivariate Gaussian distribution $X \sim \mathcal{N}(\mu, \Sigma)$, i.e.,



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$$f(x) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right).$$

- $\mu = (\mu^1, \ldots, \mu^n) \in \mathbb{R}^n$: **mean vector**;
- $\Sigma = (\sigma_{ij})_{i,j} \in \mathbb{R}^{n \times n}$: **covariance matrix**, i.e., symmetric positive semidefinite matrix.

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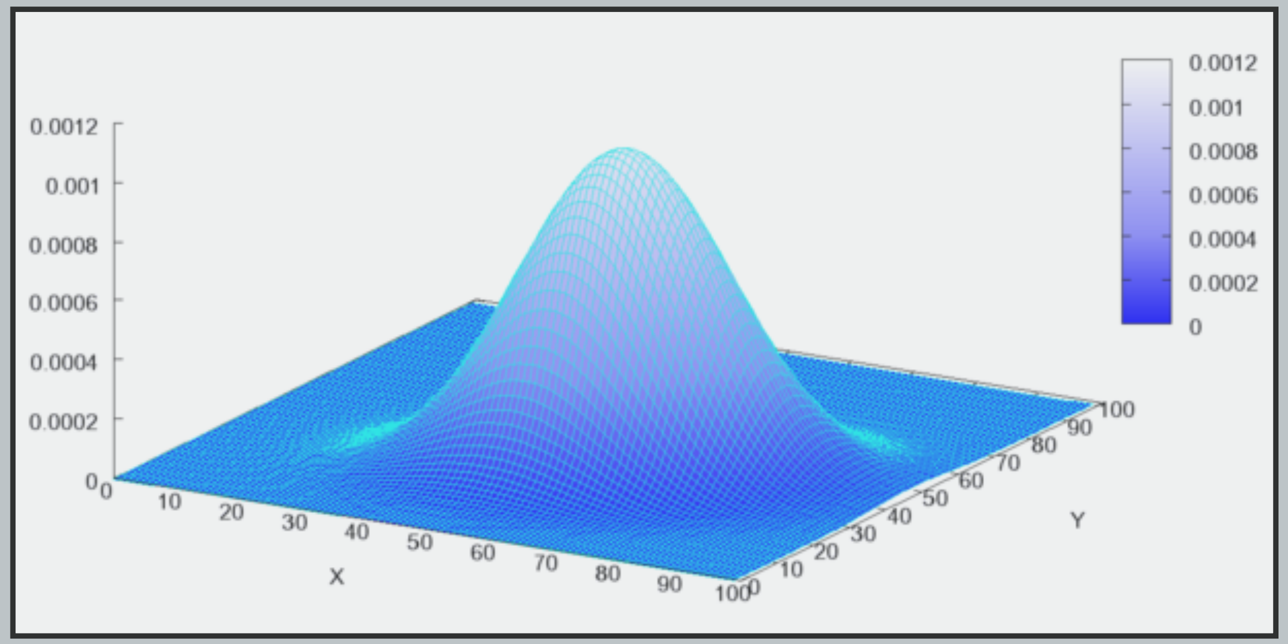

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They define the homogeneous polynomials:

$$\ell = \mu^T X = \mu^1 Y_1 + \ldots + \mu^n Y_n; \qquad q = X^T \Sigma X = \sum_{ij} \sigma_{ij} X_i X_j.$$

Then, the **moment generating function** is

$$\mathbb{E}[\exp(\langle X, Y \rangle)] = \sum_d \frac{1}{d!} m_d = \exp\left(\ell + \frac{1}{2}q\right) = \sum_d \frac{1}{d!}\left(\ell + \frac{1}{2}q\right)^d.$$

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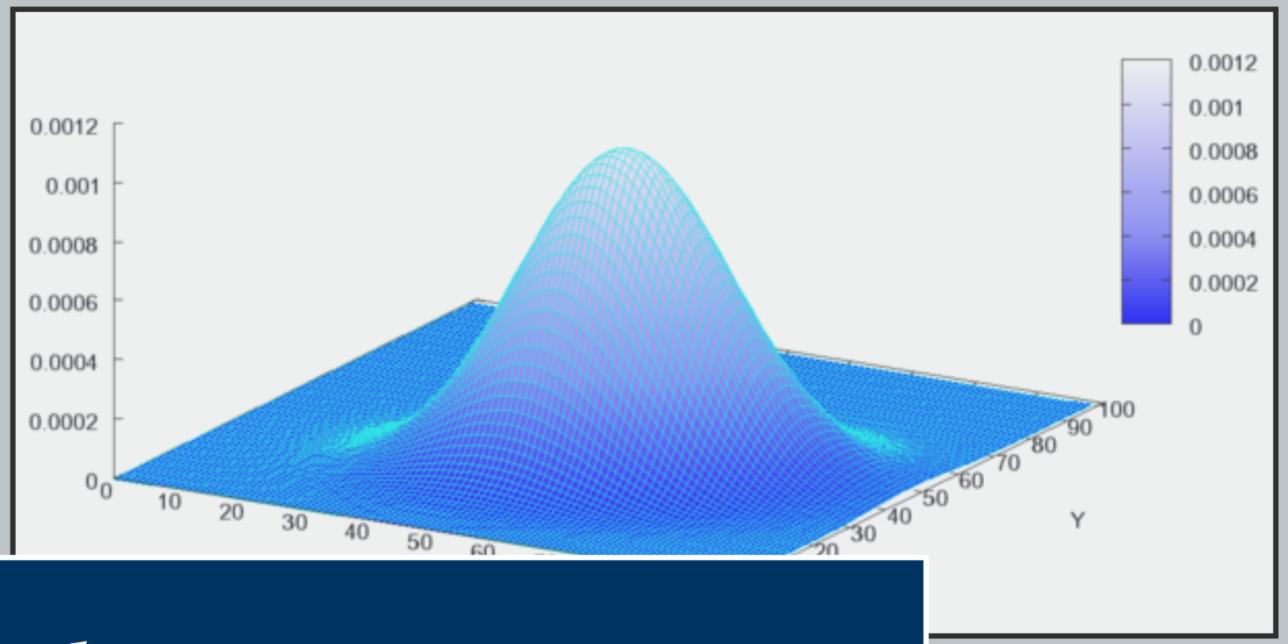

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Moments of mixtures of Gaussian distributions

Consider a multivariate Gaussian distributions

$$X_1 \sim \mathcal{N}(\mu_1, \Sigma_1), \ldots, X_m \sim \mathcal{N}(\mu_m, \Sigma_m),$$

and consider a **mixture**

$$X = \lambda_1 X_1 + \ldots + \lambda_m X_m, \qquad \text{where } \lambda_i \in (0,1), \ \sum_i \lambda_i = 1.$$

Then, the moment generating function is

$$\mathbb{E}[\exp(\langle X, Y \rangle)] = \sum_{i=1}^{m} \lambda_i \mathbb{E}[\exp(\langle X_i, Y \rangle)]$$

$$= \exp\left(\lambda_1 \mu_1 + \ldots + \lambda_m \mu_m\right) \exp\left(\frac{1}{2}(\lambda_1 q_1 + \ldots + \lambda_m q_m)\right) = \sum_d \frac{1}{d!} \sum_{i=1}^{m} \lambda_i \left(\ell_i + \frac{1}{2} q_i\right)^d$$

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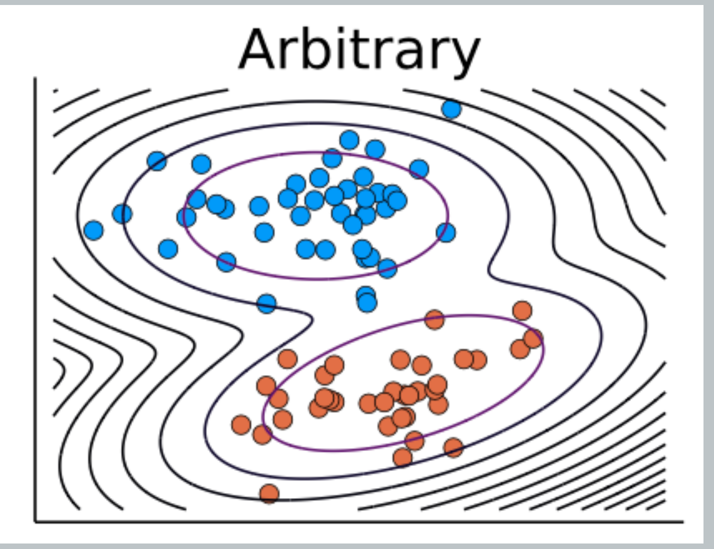
Identifiability of mixtures of Gaussian distributions

Consider a multivariate Gaussian distributions



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For some $d \in \mathbb{N}$, consider the **inverse problem** of the map sending the parameters of the model to finitely-many moments:

$$(\lambda_1, \ldots, \lambda_m, \ell_1, \ldots, \ell_m, q_1, \ldots, q_m) \mapsto (m_1, m_2, \ldots, m_d).$$

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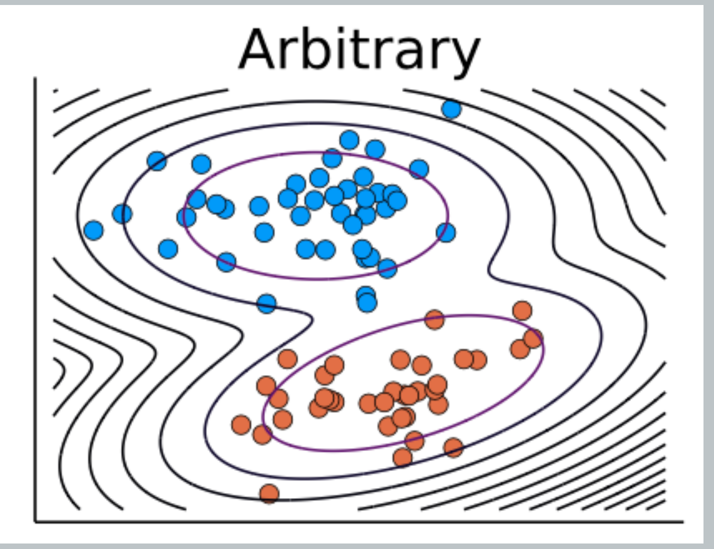

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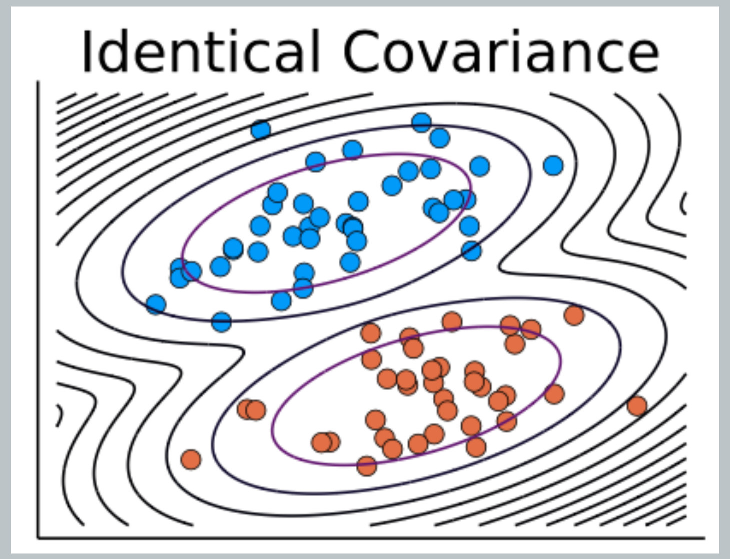
Question. Given finitely-many moments $(m_1, m_2, \ldots, m_d)$, are the parameters of the mixture model *generically* identified (up to permutation)?

Identifiability of mixtures of Gaussian distributions

the case with identical covariance

Consider a multivariate Gaussian distributions



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If $\ell_i = x^T \mu_i$ and $q = x^T \Sigma x$, then we have that the first moments are

$$m_1 = \sum_{i=1}^m \ell_i, \quad m_2 = \sum_{i=1}^m \ell_i^2 + mq, \quad m_3 = \sum_{i=1}^m \ell_i^3 + 3mq \sum_{i=1}^m \ell_i .$$

A. Taveira Blomenhofer. *Gaussian Mixture Separation and Denoising on Parameterized Varieties.* PhD thesis, U. Konstanz. 2022

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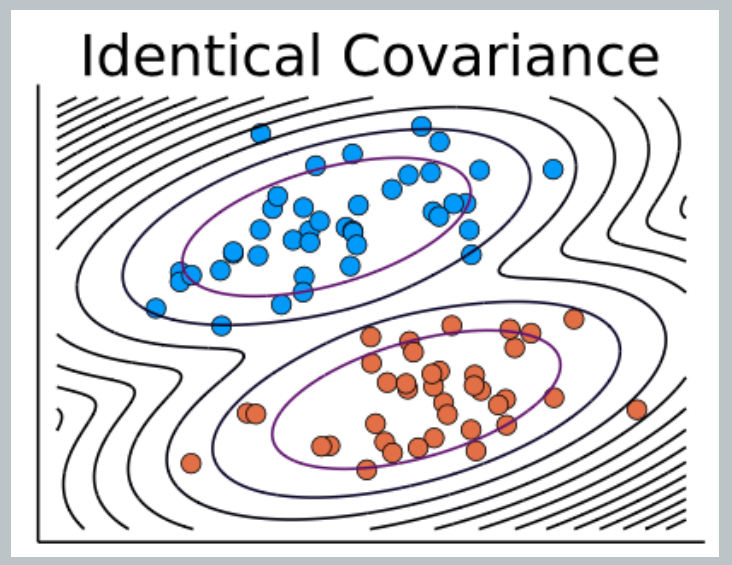

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By shifting the space, we may assume $\sum_{i=1}^m \ell_i = 0$.

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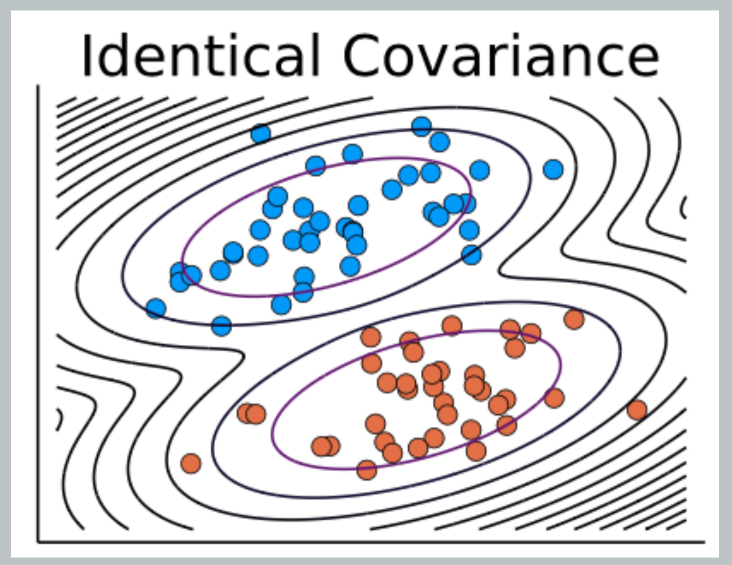

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Question. Is $m_3 = \sum_{i=1}^m \ell_i^3$ generically identifiable? I.e., given a general cubic homogeneous polynomial, is its expression as sum of cubes unique up to permutation?

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Question. Is the map $m_3 = \sum_{i=1}^{m} \ell_i^3$ generically identifiable? I.e., given a general cubic homogeneous polynomial, is its expression as sum of cubes unique up to permutation?

Given a homogeneous polynomial $f \in \mathrm{Sym}^d(\mathbb{C}^n)$, its **Waring rank** is

$$\mathrm{rk}(f) = \min_m \left\{ f = \sum_{i=1}^{m} \ell_i^d \ : \ \ell_i \in \mathrm{Sym}^1(\mathbb{C}^n) \right\}.$$

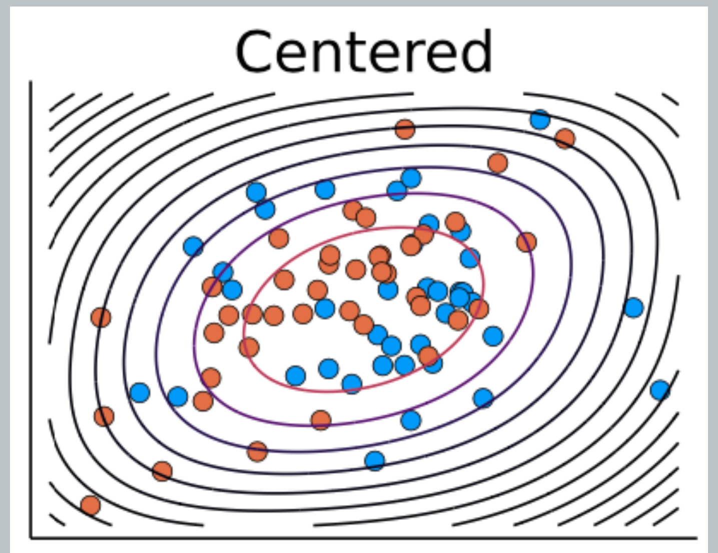
Theorem (Chiantini-Ottaviani-Vannieuwenhoven, 2017). For $d \geq 3$, the general homogeneous polynomial $f \in \mathrm{Sym}^d(\mathbb{C}^n)$ of Waring rank smaller than the generic one has a unique minimal Waring decomposition except for $(d, n, m) \in \{(6,2,9), (4,3,8), (3,5,9)\}$.

L. Chiantini, G. Ottaviani, N. Vannieuwenhoven. *On generic identifiability of symmetric tensors of subgeneric ranks.* Transactions of AMS. 2017

Identifiability of mixtures of Gaussian distributions

the centered case

Consider a multivariate Gaussian distributions



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$$m_i = \begin{cases} 0 & \text{for } i = 2d+1 \\ q_1^d + \ldots + q_m^d & \text{for } i = 2d \end{cases}.$$

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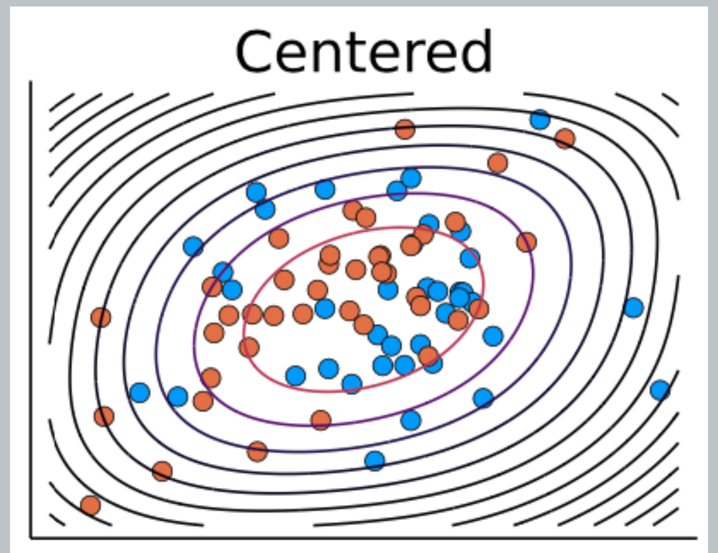

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I.e., given a general homogeneous polynomial of even degree, is its expression as sum of powers of quadrics unique up to permutation?

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Observation. For degree-2 and degree-4 moments we cannot have identifiability. Indeed:

- $q_1 + q_2 = (q_1 + q') + (q_2 - q')$, for any quadric q';
- $q_1^2 + q_2^2 = \frac{1}{2}(q_1 + q_2)^2 + \frac{1}{2}(q_1 - q_2)^2$.

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Identifiability of mixtures of Gaussian distributions

the centered case

Theorem (Taveira Blomenhofer-Casarotti-Michałek-Oneto, 2022*). For $m, n \in \mathbb{N}$ such that

$$m \leq \min\left\{ \frac{\binom{n+5}{6}}{\binom{n+1}{2}} - \binom{n+1}{2} - 1, \binom{n+1}{2} + 1 \right\}$$

the general sextic $f = \sum_{i=1}^{m} q_i^3$ is identifiable.

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Corollary (Taveira Blomenhofer-Casarotti-Michałek-Oneto, 2022*).
For $m, n \in \mathbb{N}$ such that the Theorem holds then:

1. the uniformly weighted Gaussian mixture $X = \frac{1}{m}(X_1 + \ldots + X_m)$ is generically identified by its degree-6 moment;
2. the general Gaussian mixture $X = \lambda_1 X_1 + \ldots + \lambda_m X_m$ is generically identified by its degree-6 and degree-4 moments.

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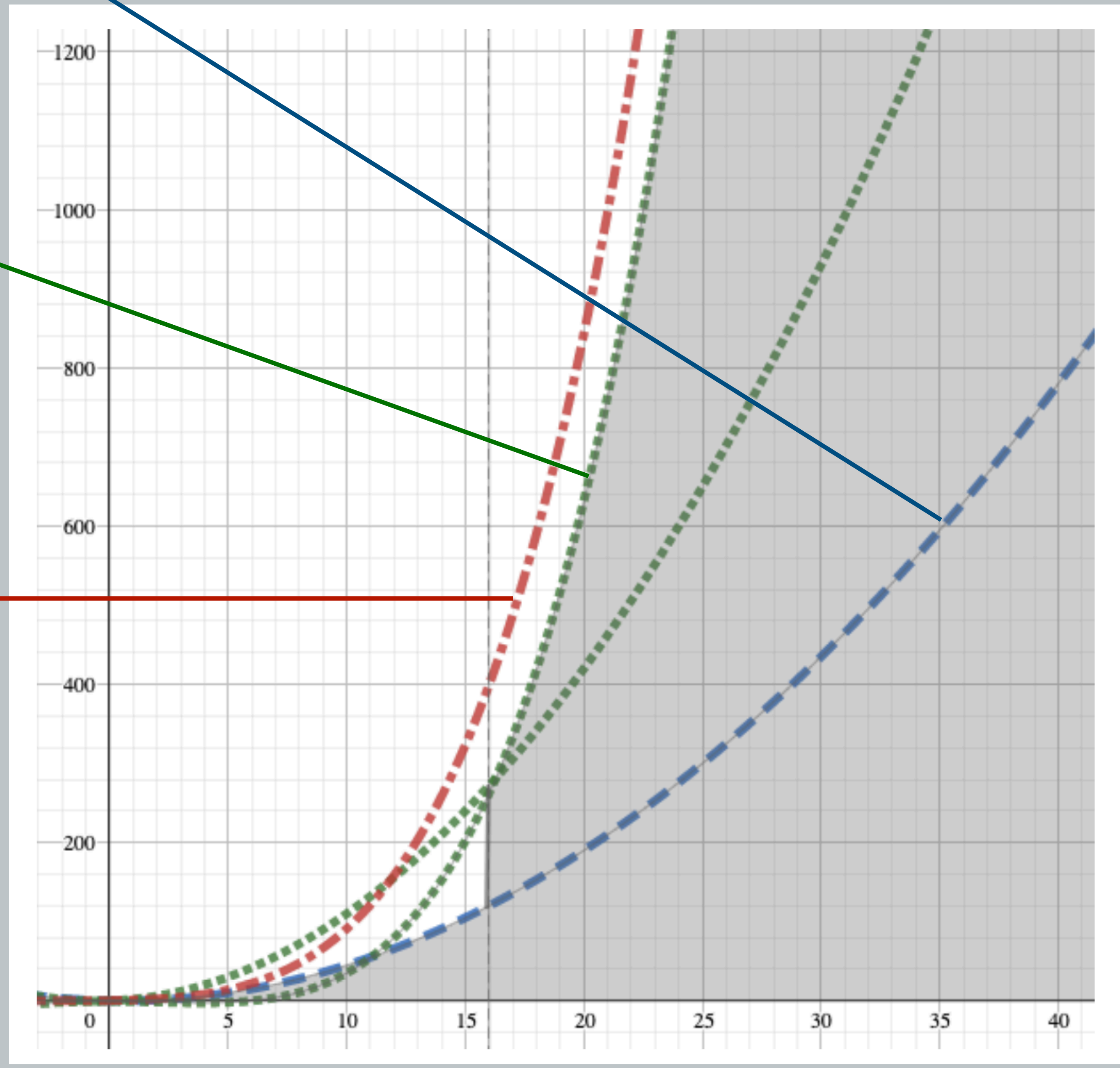
$$\frac{\binom{n+5}{6}}{\binom{n+1}{2}}$$



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From non-defectivity to identifiability

Given any algebraic variety $X \subset \mathbb{P}^N$, we define:

- For any point $q \in \mathbb{P}^N$, the **X-rank** of q is $\mathrm{rk}_X(q) = \min_m\{q \in \langle p_1, \ldots, p_m \rangle \ : \ p_i \in X\}$.
- The **m-secant variety** of X is $\sigma_m(X) = \overline{\bigcup_{p_1,\ldots,p_m \in X} \langle p_1, \ldots, p_m \rangle} = \overline{\{q \ : \ \mathrm{rk}_X(q) \leq m\}}$.

A. Casarotti, M. Mella. *From non-defectivity to identifiability.* Journal of the EMS. 2020

From non-defectivity to identifiability

Given any algebraic variety $X \subset \mathbb{P}^N$, we define:

- For any point $q \in \mathbb{P}^N$, the **X-rank** of q is $\mathrm{rk}_X(q) = \min_m \{q \in \langle p_1, \ldots, p_m \rangle \ : \ p_i \in X\}$.
- The **m-secant variety** of X is $\sigma_m(X) = \overline{\bigcup_{p_1,\ldots,p_m \in X} \langle p_1, \ldots, p_m \rangle} = \overline{\{q \ : \ \mathrm{rk}_X(q) \leq m\}}$.

The variety X is said **m-defective** if $\dim \sigma_m(X) < \min\{N, m \dim(X) + m - 1\}$.

The variety X is said **m-identifiable** if the general element $q \in \sigma_m(X)$ is identifiable.

A. Casarotti, M. Mella. *From non-defectivity to identifiability.* Journal of the EMS. 2020

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Theorem (Casarotti-Mella, 2020). If X is smooth, $m > 2\dim(X)$, $m \dim(X) + m - 1 < N$, then non m-defectivity implies $(m-1)$-identifiability.

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Theorem (Massarenti-Mella, 2022). If X is smooth, $\cancel{m > 2\dim(X)}$, $m\dim(X) + m - 1 < N$, then non m-defectivity implies $(m-1)$-identifiability.

A. Massarenti, M. Mella. *Bronowski's Conjecture and identifiability of projective varieties.* arXiv:2210.13524. 2022

Identifiability of mixtures of Gaussian distributions

the centered case

Theorem (Taveira Blomenhofer-Casarotti-Michałek-Oneto, 2022*). For $m, n \in \mathbb{N}$ such that

$$m \leq \min\left\{ \underline{\frac{\binom{n+5}{6}}{\binom{n+1}{2}} - \binom{n+1}{2} - 1}, \underline{\binom{n+1}{2} + 1} \right\}$$

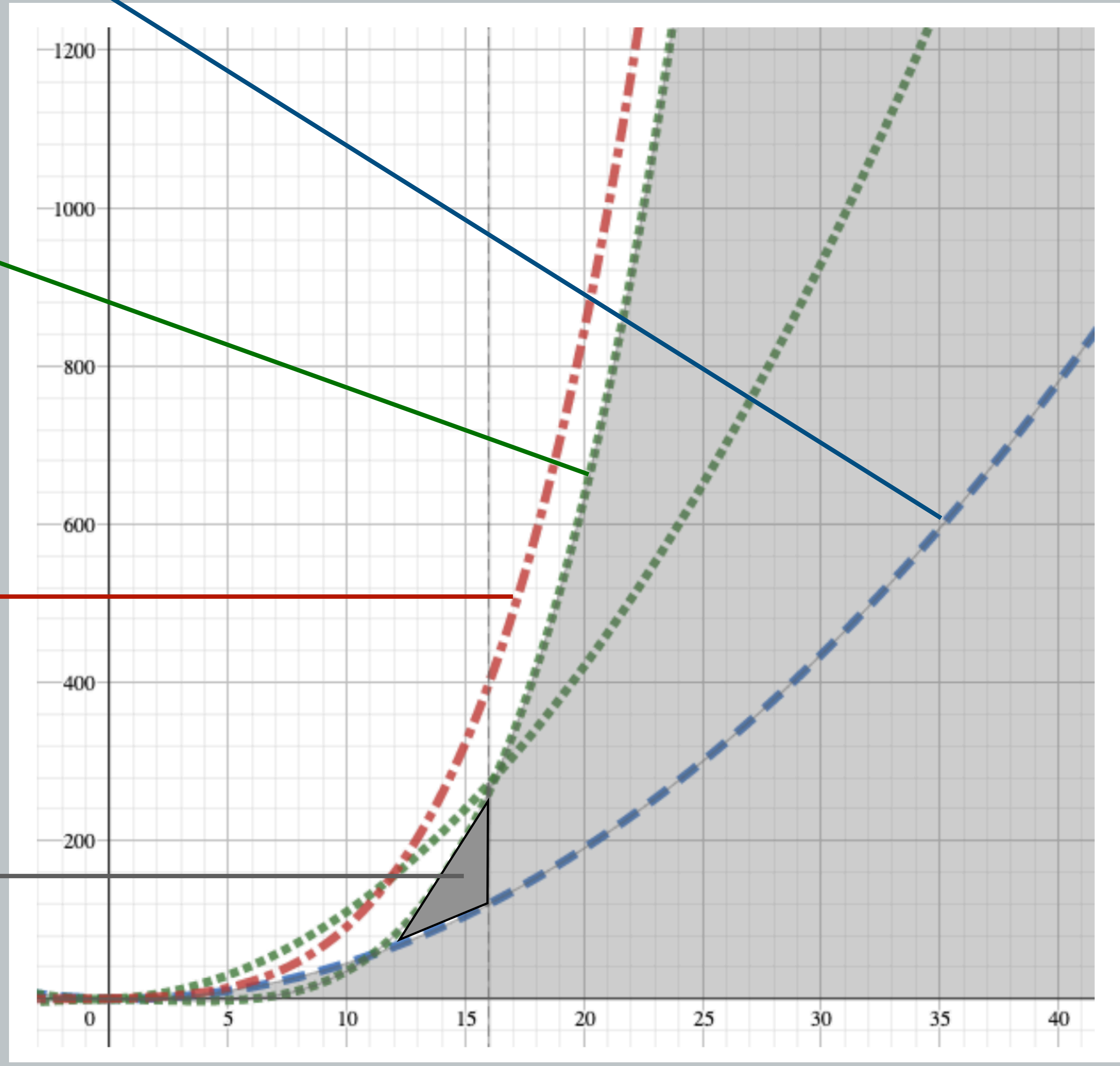
the general sextic $f = \sum_{i=1}^{m} q_i^3$ is identifiable.

from non-defectivity
to identifiability

tangential contact locus

$\dfrac{\binom{n+5}{6}}{\binom{n+1}{2}}$ the expected number of cubes needed by the general sextic

Massarenti-Mella, 2022



A. Taveira Blomenhofer, A. Casarotti, M. Michałek, A. Oneto. *Identifiability for mixtures of centered Gaussians and sums of powers of quadratics.* arXiv:2204.09356v2. 2022

Non-defectivity of sums of powers

and Fröberg's Conjecture

Let $V_{d,k,n} = \left\{[g^k] \ : \ g \in \mathbb{P}\mathrm{Sym}^d(\mathbb{C}^n)\right\} \subset \mathbb{P}\mathrm{Sym}^{dk}(\mathbb{C}^n)$

$$
\begin{tikzcd}
\mathbb{P}\mathrm{Sym}^d(\mathbb{C}^n) \arrow[r] \arrow[dr] & \mathbb{P}\mathrm{Sym}^k\left(\mathrm{Sym}^d(\mathbb{C}^n)\right) \arrow[d, dashed] \\
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\end{tikzcd}
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Conjecture (Ottaviani, 2017).
For $k \geq 3,\ d \geq 2$, the variety $V_{d,k,n}$ is never m-defective.

S. Lundqvist, A. Oneto, B. Reznick, B. Shapiro. *On generic and maximal k-ranks of binary forms.* Journal of Pure and Applied Algebra. 2017

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Lemma (Terracini, 1911). Let $X \subset \mathbb{P}^N$ smooth.
Given general points $p_1, \ldots, p_m \in X$ and general $q \in \langle p_1, \ldots, p_m \rangle$, then

$$T_q\sigma_m(X) = \langle T_{p_1}(X), \ldots, T_{p_m}(X) \rangle.$$

S. Lundqvist, A. Oneto, B. Reznick, B. Shapiro. *On generic and maximal k-ranks of binary forms.* Journal of Pure and Applied Algebra. 2017

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Note that $\frac{d}{dt}|_{t=0}(g+th)^k = kg^{k-1}h$.

Given general forms $g_1, \ldots, g_m \in \mathrm{Sym}^d(\mathbb{C}^n)$and $I = (g_1^{k-1}, \ldots, g_m^{k-1})$,
the **general tangent space** to $\sigma_m(V_{d,k,n})$ is

$$\mathbb{P}\left(I \cap \mathrm{Sym}^{dk}(\mathbb{C}^n)\right) = \mathbb{P}[(g_1^{k-1}, \ldots, g_m^{k-1})]_{kd}$$

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Non-defectivity of sums of powers

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Conjecture (Nicklasson, 2017).
Given general forms $g_1, \ldots, g_m \in \mathrm{Sym}^d(\mathbb{C}^n)$ and $I = (g_1^{k-1}, \ldots, g_m^{k-1})$,

$$\textstyle\sum_i \dim\left(I \cap \mathrm{Sym}^i(\mathbb{C}^n)\right) t^i = \left[\frac{(1-t^{d(k-1)})^m}{(1-t)^n}\right],$$

where $[\,\cdot\,]$ means to truncate the power series before the first non-positive coefficient.
I.e., the ideal I has the minimal Hilbert function prescribed by Fröberg's Conjecture.

R. Fröberg. *An inequality for Hilbert series of graded algebras*. Mathematica Scandinavica. 1985
A. Oneto, *Waring-type problems for polynomials.* PhD thesis, Stockholm U., 2016
L. Nicklasson. *On the Hilbert series of ideals generated by generic forms.* Communications in Algebra. 2017
A note on Fröberg's conjecture for forms of equal degrees. *Comptes Rendus Mathematique.* 2017

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$$\mathrm{coeff}_{kd}\frac{(1-t^{d(k-1)})^m}{(1-t)^n} = \binom{n+dk-1}{dk} - m\binom{n+d-1}{d} = exp\,.\,\mathrm{codim}\sigma_m(V_{d,k,n})$$

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Theorem (Nenashev, 2015). If $m \leq \dfrac{\dim \mathrm{Sym}^{d(k-1)}(\mathbb{C}^n)}{\dim \mathrm{Sym}^d(\mathbb{C}^n)} - \dim \mathrm{Sym}^d(\mathbb{C}^n)$,
then the Conjecture holds.

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Thank you!

A. Taveira Blomenhofer, A. Casarotti, M. Michałek, **A. Oneto**.
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