Secant defectivity of Segre-Veronese varieties via collapsing points

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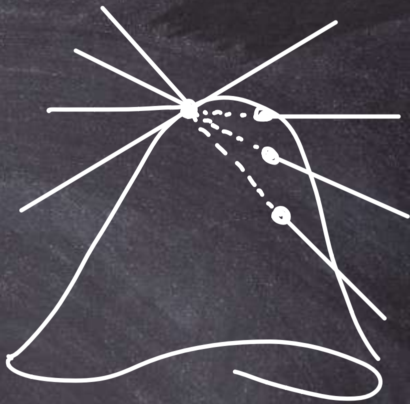
Let X be a projective variety in $\mathbb{P}^N$.



The r-th secant variety of X is

$$\sigma_r(X) = \overline{\bigcup_{p_1 \dots p_r \in X} \langle p_1, \dots, p_r \rangle} \subseteq \mathbb{P}^N$$

The expected dimension is

$$\text{exp. dim } \sigma_r(X) = \min\{N, r\dim(X) + r - 1\}$$

X is r-defective if $\dim \sigma_r(X) < \text{exp. dim } \sigma_r(X)$.

$$\boxed{\begin{array}{c}\text{Question (since late XIX century)} \\ \text{PROVIDE CLASSIFICATIONS OF DEFECTIVE VARIETIES}\end{array}}$$

d-th Veronese embedding

$$\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$$

$$(a_0 : \dots : a_n) \longrightarrow (a_0^d : a_0^{d-1} a_1 : \dots : a_n^d)$$

$N = \binom{n+d}{d} - 1$

Alexander-Hirschowitz (1995)

The Veronese variety $\nu_d(\mathbb{P}^n)$ is r-defective if and only if

(1) $d=2$, $2 \le r \le n$

(2) $d=4$, $n=2$, $r=5$

(3) $d=4$, $n=3$, $r=9$

(4) $d=3$, $n=4$, $r=7$

(5) $d=4$, $n=4$, $r=14$

Segre-Veronese embedding (on two factors)

$$\nu_{d,e} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^N \qquad N = \binom{m+d}{m}\binom{n+e}{n} - 1$$

$$((a_0 : \ldots : a_m), (b_0 : \ldots : b_n)) \longmapsto (a_0^d b_0^e : a_0^{d-1} a_1 b_0^e : \ldots : a_m^d b_n^e)$$

A list of defective cases :

(m,n)	(d,e)	r	ref.
$(2,2k+1)$	$(1,2)$	$3k+2$	Ottaviani '08
$(4,3)$	$(1,2)$	6	Carlini-Chipalkatti '03
$(1,2)$	$(1,3)$	5	Dionisi-Fontanari '01
$(1,n)$	$(2,2)$	$n+2 \leq r \leq 2n+1$	Catalisano-Geramita-Gimigliano '05
$(2,2)$	$(2,2)$	$7,8$	—— " ——
$(2,n)$	$(2,2)$	$\lfloor \frac{n^2+9n+5}{n+3} \rfloor \leq r \leq 3n+2$	—— " ——
$(3,3)$	$(2,2)$	$14,15$	—— " ——
$(3,4)$	$(2,2)$	19	Bocci '05
$(n,1)$	$(2,2k)$	$kn+k+1 \leq r \leq kn+k+n$	Abrescia '08

Segre-Veronese embedding (on two factors)

Conjecture (Abo-Brambilla, 2013)

If $d,e \geq 3$, then $\nu_{d,e}(\mathbb{P}^m \times \mathbb{P}^n)$ is never defective.

Theorem (Abo-Brambilla, 2013)

If the statement is true for $(d,e) \in \{(3,3),(3,4),(4,4)\}$, then the conjecture holds.

Segre-Veronese embedding (on two factors)

THEOREM

If $d, e \geq 3$, then $\nu_{d,e}(\mathbb{P}^m \times \mathbb{P}^n)$ is never defective.

Theorem (Abo-Brambilla, 2013)

If the statement is true for $(d,e) \in \{(3,3),(3,4),(4,4)\}$, then the conjecture holds.

THEOREM (GALUPPI-ONETO, arXiv: 2104.02522)

If $(d,e) \in \{(3,3),(3,4),(4,4)\}$, then

$$\nu_{d,e}(\mathbb{P}^m \times \mathbb{P}^n) \text{ is never defective}$$

WHY ?

Varieties parametrized by rank-one tensors

Segre varieties : general case

$$\nu_{1,\dots,1} : \mathbb{P}V_1 \times \dots \times \mathbb{P}V_m \longrightarrow \mathbb{P}\, V_1 \otimes \dots \otimes V_m$$

$$(v_1, \dots, v_m) \longmapsto v_1 \otimes \dots \otimes v_m$$

Veronese varieties : symmetric case

$$\nu_d : \mathbb{P}V \longrightarrow \mathbb{P}\,\mathrm{Sym}^d V$$

$$v \longmapsto v^{\otimes d}$$

Segre-Veronese varieties : partially symmetric case

$$\nu_{d_1,\dots,d_m} : \mathbb{P}V_1 \times \dots \times \mathbb{P}V_m \longrightarrow \mathbb{P}\,\mathrm{Sym}^{d_1} V_1 \otimes \dots \otimes \mathrm{Sym}^{d_m} V_m$$

$$(v_1, \dots, v_m) \longmapsto v_1^{\otimes d_1} \otimes \dots \otimes v_m^{\otimes d_m}$$

WHY ?

Segre-Veronese varieties are parametrized by partially symmetric tensors of rank-one

$$\nu_{d,e} : \mathbb{P}V_1 \times \mathbb{P}V_2 \longrightarrow \mathbb{P}\,\mathrm{Sym}^d V_1 \otimes \mathrm{Sym}^e V$$

$$(v_1, v_2) \longmapsto v_1^{\otimes d} \otimes v_2^{\otimes e}$$

and their r-th secant varieties are the closure of set of tensors of rank at most r

$$\mathrm{rk}(T) = \min \left\{ r \mid T = \sum_{i=1}^{r} v_1^{\otimes d} \otimes v_2^{\otimes e} \right\}$$

$$\sigma_r\left(\nu_{d,e}(\mathbb{P}V_1 \times \mathbb{P}V_2)\right) = \overline{\left\{ T \mid \mathrm{rk}(T) \leq r \right\}}$$

WHY ?

Segre-Veronese varieties are parametrized by partially symmetric tensors of rank-one

$$\begin{array}{rccc} \nu_{d,e} : & \mathbb{P}V_1 \times \mathbb{P}V_2 & \longrightarrow & \mathbb{P}\,\mathrm{Sym}^d V_1 \otimes \mathrm{Sym}^e V \\ & (v_1, v_2) & \longmapsto & v_1^{\otimes d} \otimes v_2^{\otimes e} \end{array}$$

and their r-th secant varieties are the closure of set of tensors of rank at most r

$$\mathrm{rk}(T) = \min \left\{ r \;\middle|\; T = \sum_{i=1}^{r} v_1^{\otimes d} \otimes v_2^{\otimes e} \right\}$$

If T general,

$$\mathrm{rk}(T) = \min \{ r \mid \sigma_r(\nu_{d,e}(\mathbb{P}V_1 \times \mathbb{P}V_2)) = \mathbb{P}\,\mathrm{Sym}^d V_1 \otimes \mathrm{Sym}^e V_2 \}$$

How ?

Computing dimensions of secant varieties is an interpolation problem

Terracini's Lemma (1911)

Let $p_1, \dots, p_r \in X$ general points, $p \in \langle p_1, \dots, p_r \rangle$ general point. Then,

$$T_p \sigma_r X = \langle T_{p_1} X, \dots, T_{p_r} X \rangle$$

$$\operatorname{codim} \sigma_r X = \dim \{ H \text{ hyperplanes} \mid H \supseteq T_{p_i} X \ \forall i \in \{1 \dots r\} \}$$

If $X = \nu_{d_1, d_2}(\mathbb{P}^m \times \mathbb{P}^n)$, then by pulling-back :

$$= \dim \left\{ D \subseteq \mathbb{P}^m \times \mathbb{P}^n \;\middle|\; \begin{array}{l} \deg(D) = (d,e) \\ 2p_i \in D \ \forall i \in \{1 .. r\} \end{array} \right\}$$

How ?

Computing dimensions of secant varieties is an interpolation problem

$$\operatorname{codim} \sigma_r \nu_{d,e} \mathbb{P}^m \times \mathbb{P}^n = \dim \underbrace{\mathcal{L}^{d,e}_{m \times n}(2)}$$

linear system of degree-(d,e) divisors of $\mathbb{P}^m \times \mathbb{P}^n$ through a general scheme of r 2-fat points with

$$I(2) = \wp_1^2 \cap \dots \cap \wp_r^2$$

$$= \dim I(2)_{d,e}$$

How ?

Computing dimensions of secant varieties is an interpolation problem

$$\text{exp.codim}\ \sigma_r \nu_{d,e} \mathbb{P}^m \times \mathbb{P}^n \leq \text{codim}\ \sigma_r \nu_{d,e} \mathbb{P}^m \times \mathbb{P}^n$$

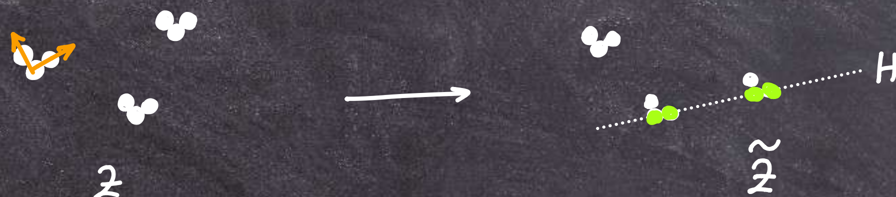
$$= \dim \mathcal{L}^{d,e}_{m\times n}(Z) \leq \underline{\dim \mathcal{L}^{d,e}_{m\times n}(\tilde{Z})}$$

$\tilde{Z}$ is a degeneration of Z

Goal : find a degeneration to prove that this is actually a chain of equalities

How ?

(Classically) $\tilde{Z}$ has some of the supports on a hyperplane



(Castelnuovo's exact sequence)

$$\boxed{0 \longrightarrow \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}+H) \longrightarrow \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}) \longrightarrow \mathcal{L}^{d,e}_{H}(Z\cap H)}$$

If $H \cong \mathbb{P}^{m-1}\times\mathbb{P}^{n}$ is a divisor of bidegree $(1,0)$:

$$0 \longrightarrow \mathcal{L}^{d-1,e}_{m\times n}(\tilde{Z}) \longrightarrow \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}) \longrightarrow \mathcal{L}^{d,e}_{m-1\times n}(Z\cap H)$$

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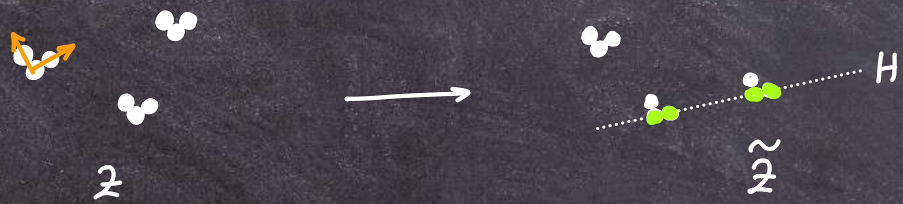


(Castelnuovo's exact sequence)

$$0 \longrightarrow \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}+H) \longrightarrow \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}) \longrightarrow \mathcal{L}^{d,e}_{H}(Z\cap H)$$

$$\dim \mathcal{L}^{d,e}_{m\times n}(\tilde{Z}) \leq \dim \mathcal{L}^{d-1,e}_{m\times n}(\tilde{Z}) + \dim \mathcal{L}^{d,e}_{m-1\times n}(\tilde{Z}\cap H)$$

How ?

The classical strategy has some arithmetical constraint and cannot work in general.

Example (Sextics of $\mathbb{P}^3$ through 21 2-fat points)

$$\text{exp.dim}\ \mathcal{L}_3^6(2^{21}) = \binom{6+3}{3} - 21\cdot 4 = 84 - 84 = 0$$

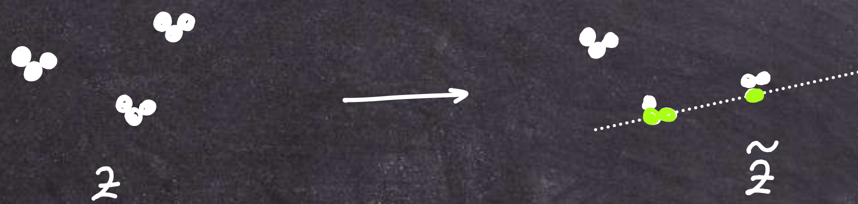
Parameter count for $\mathcal{L}_2^6(2^r)$:

$$\binom{2+6}{2} - 3r = 28 - 3r \neq 0$$

How ?

The classical strategy has some arithmetical constraint and cannot work in general.

(Alexander-Hirschowitz, ~1980s) – differential Horace method

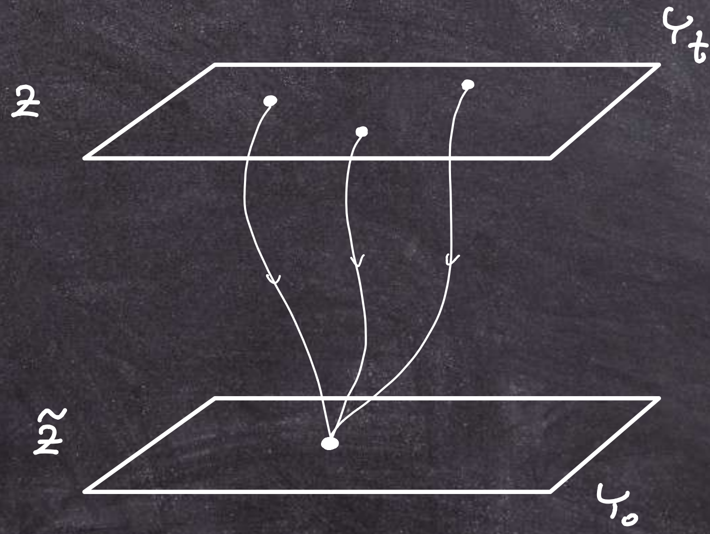


↳ Residual schemes might be too complicated to be treated in general

... but still, very succesfull

How ?

(Evain, 1997) Some of the components of Z are collapsed together

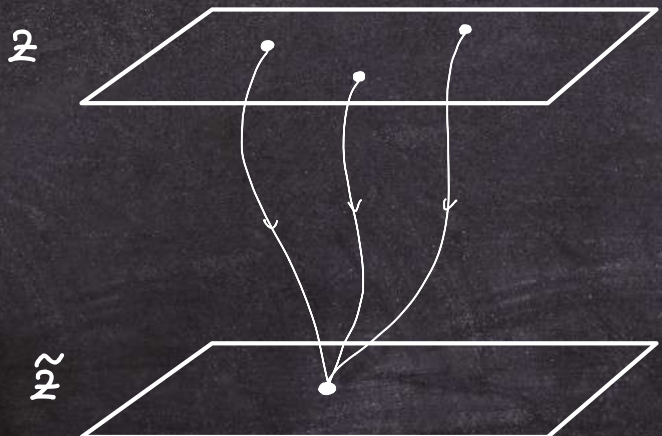


COLLAPSING n+1 2-FAT POINTS

X : n-dimensional smooth variety ;

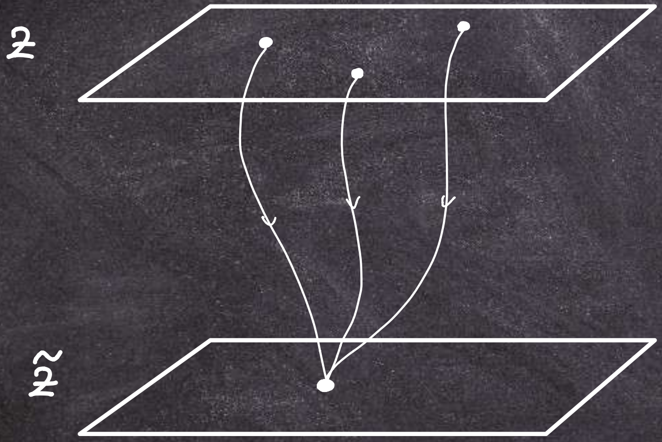
Z : scheme of $n+1$ 2-fat points on X

$\tilde{Z}$: result of a general degeneration of Z to a unique support.



Then $\tilde{Z}$ is :

3-fat point with $\binom{n+1}{2}$ extra points infinitesimaly close



Then $\tilde{Z}$ is :

3-fat point with $\binom{n+1}{2}$ extra points infinitesimaly close

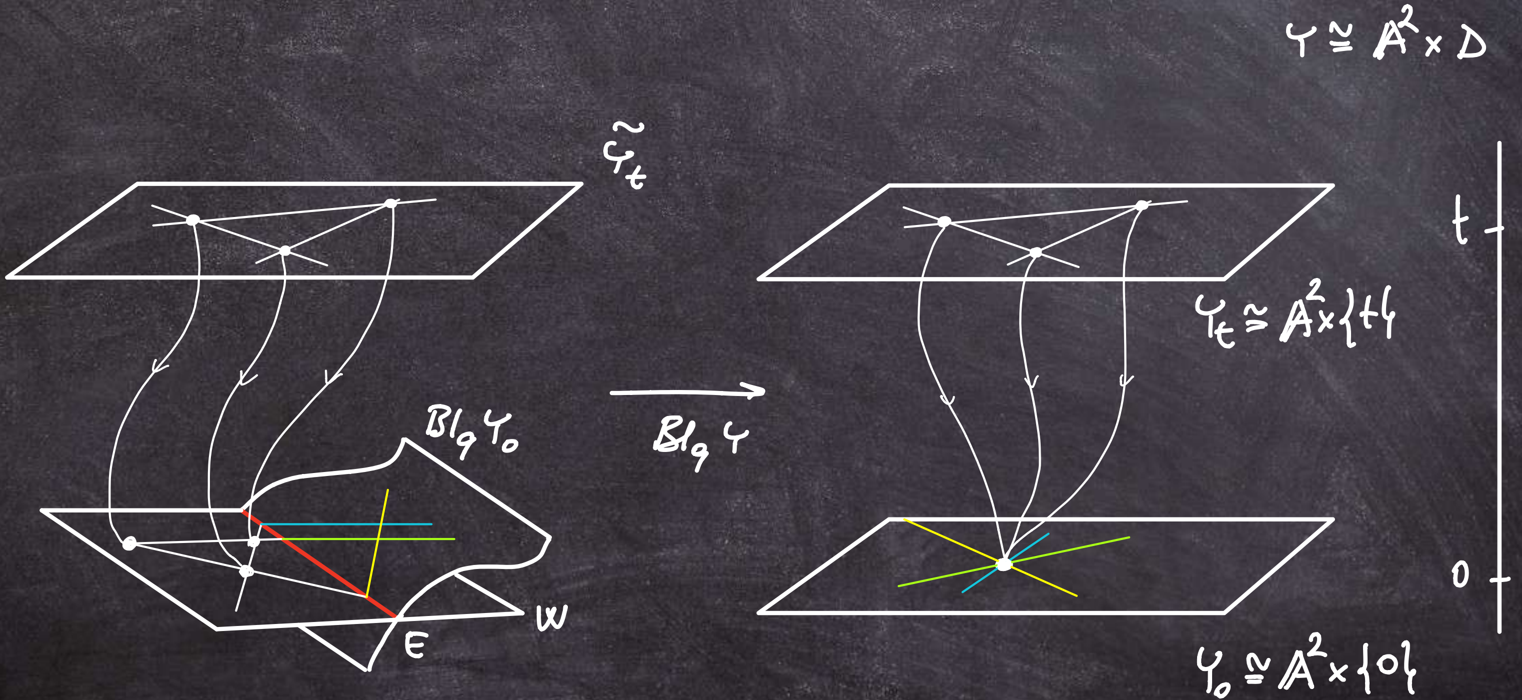
Equivalently: if H is a line throug the support of $\tilde{Z}$,

$\tilde{Z} \cap H$ is a 3-fat point on the line H,

except for $\binom{n+1}{2}$ line for which it is a 4-fat point.

```
i1 : S = QQ[t][x,y,z];

i2 : I = trim intersect(
         (ideal(x,y-t*z))^2,
         (ideal(x-t*z,y))^2,
         (ideal(x-t*z,y-t*z))^2
         );

o2 : Ideal of S

i3 : I0 = sub(I,t=>0);

o3 : Ideal of S

i4 : -- The scheme contains the 3-fat point
     isSubset(I0, (ideal(x,y))^3)

o4 = true

i5 : -- It has 3 extra points infinitesimally close
     degree(I0)-degree((ideal(x,y))^3)

o5 = 3

i6 : -- The intersection with a general line is a 3-fat point on the line..
     degree(I0 + ideal(x*random(QQ)+y*random(QQ)))

o6 = 3

i7 : -- ..but there are three special lines for which it is a 4-fat point on the line.
     degree(I0 + ideal(x+y))

o7 = 4

i8 : degree(I0 + ideal(x))

o8 = 4

i9 : degree(I0 + ideal(y))

o9 = 4
```

A geometric intuition of this construction can be explained by the following picture.



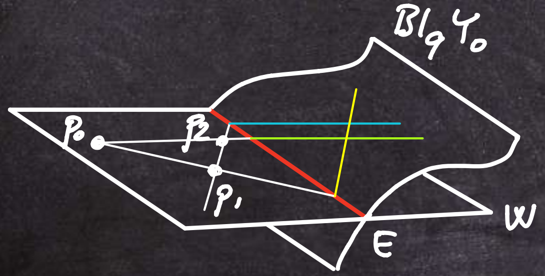
This construction generalize in higher dimension.

If $p_0, \dots, p_n$ are the points on the exceptional divisor $W \cong \mathbb{P}^n$,

then, the $\binom{n+1}{2}$ extra direction in $\tilde{Z}$

are given by the points in $E \cong \mathbb{P}^{n-1}$: $t_{i,j} = \langle p_i, p_j \rangle \cap E$

Let $T = \{ t_{i,j} \}$.



These are not general, but it is possible to show that

$\mathcal{L}^2_E(T)$, $\mathcal{L}^3_E(T)$, $\mathcal{L}^3_E(2T)$

have the expected dimension

IDEA OF PROOF OF MAIN RESULT

Let Z be a scheme of r general 2-fat points in $\mathbb{P}^m \times \mathbb{P}^n$

We want to show that for every r

$$\dim \mathcal{L}^{d,e}_{m\times n}(Z) = \dim \mathcal{L}^{d,e}_{m\times n}(2^r)$$

$$= \max\left\{0, \binom{m+d}{m}\binom{n+e}{n} - r(m+n+1)\right\}$$

- We restrict to $r \in \left\{ \left\lfloor \frac{\binom{m+d}{m}\binom{n+e}{n}}{m+n+1} \right\rfloor, \left\lceil \frac{\binom{m+d}{m}\binom{n+e}{n}}{m+n+1} \right\rceil \right\} = \{r_*, r^*\}$
    - If Z impose independent conditions, so does $Z' \subseteq Z$
    - If there are no divisors through Z, so is for $Z' \supseteq Z$

IDEA OF PROOF OF MAIN RESULT

Let Z be a scheme of r general 2-fat points in $\mathbb{P}^m \times \mathbb{P}^n$ for $r \in \{r_*, r^*\}$.

- since $d, e \geq 3$, $r_*, r^* \geq N+1 := m+n+1$

- Let $\tilde{Z}$ the degeneration of Z obtained by collapsing $N+1$ of the 2-fat points.

we write that $\tilde{Z}$ is of type $(3[T], 2^{r-(N+1)})$

and let q the support of the collapsed component

Assumption 1 : $\mathcal{L}^{d,e}_{m\times n}(3, 2^{r-(N+1)})$ has the expected dimension

Assuming by contradiction that $\mathcal{L}^{d,e}_{m\times n}(3[T], 2^{r-(N+1)})$ is defective

we show that:

$$\mathcal{L}^{d,e}_{m\times n}(3[T], 2^{r-(N+1)}) \subseteq \mathcal{L}^{d,e}_{m\times n}(4, 2^{r-(N+1)})$$

Assumption 2 : $\dim \mathcal{L}^{d,e}_{m\times n}(4, 2^{r-(N+1)}) = 0$

- if $r = r_* < r^*$: absurd because exp.dim $\mathcal{L}^{d,e}_{m\times n}(3[T], 2^{r-(N+1)}) > 0$
- if $r = r^*$: absurd because $\mathcal{L}^{d,e}_{m\times n}(3[T], 2^{r-(N+1)})$ is assumed to be defective.

Hence, everything reduces to check that

Assumption 1 : $\mathcal{L}^{d,e}_{m\times n}(3, 2^{r-(N+1)})$ has the expected dimension

Assumption 2 : $\dim \mathcal{L}^{d,e}_{m\times n}(4, 2^{r-(N+1)}) = 0$

$\hookrightarrow$ We do this for all three cases

$(d,e) \in \{(3,3), (3,4), (4,4)\}$

in order to conclude the proof.

— The missing cases —

THEOREM

If $d,e \geq 3$, then $\nu_{d,e}(\mathbb{P}^m \times \mathbb{P}^n)$ is never defective.

A list of defective cases : IS IT COMPLETE ?

(m,n)	(d,e)	r	ref.
$(2,2k+1)$	$(1,2)$	$3k+2$	Ottaviani '08
$(4,3)$	$(1,2)$	6	Carlini - Chipalkatti '03
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$(1,n)$	$(2,2)$	$n+2 \leq r \leq 2n+1$	Catalisano - Geramita - Gimigliano '05
$(2,2)$	$(2,2)$	$7,8$	—— " ——
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$(3,3)$	$(2,2)$	$14,15$	—— " ——
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$(n,1)$	$(2,2k)$	$kn+k+1 \leq r \leq kn+k+n$	Abrescia '08

— The missing cases —

THEOREM

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A list of defective cases : IS IT COMPLETE ?

Our method cannot apply because

3-fat points fail to impose
independent conditions in bi-degrees $(1,d)$

4-fat points fail to impose
independent conditions in bi-degrees $(1,d)$ and $(2,d)$

— A general result —

THEOREM $(X, \mathcal{L})$ a polarized smooth irreducible projective variety of dimension n such that $\mathcal{L}$ gives a proper closed embedding of X.

Let W be the image. Let $r \geq n+1$.

If: (1) $\dim \mathcal{L}(3, 2^{r-(n+1)}) = \dim \mathcal{L} - \binom{n+2}{2} - (r-n-1)(n+1)$

(2) $\dim \mathcal{L}(4, 2^{r-(n+1)}) = 0$

(3) $\dim \mathcal{L}(3) - \dim \mathcal{L}(4) \geq \binom{n+1}{2}$

Then, $\dim \mathcal{L}(2^r) = \max\{0, \dim \mathcal{L} - (n+1)r\}$.

I.e., W is not r-defective.

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