The strength of general forms

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joint works with A. Bik, E. Ballico, E. Ventura

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THE QUESTION: STRENGTH OF GENERAL FORMS

Let $S = \mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$

S_d: $\mathbb{C}$-vector space of degree-d homogeneous polynomials

[Ananyan-Hochster, 2020]

A strength decomposition of a homogeneous polynomial $f \in S_d$ is

$$f = g_1 h_1 + g_2 h_2 + \dots + g_r h_r \qquad 1 \leq \deg(g_i), \deg(h_i)$$

The strength of $f \in S_d$ is the smallest length of a strength decomposition.

[Ananyan-Hochster, 2020; Erman-Sam-Snowden, 2020; Bik-Draisma-Eggermont, 2019; Kazhdan-Ziegler, 2018; Derksen-Eggermont-Snowden, 2017; ...]

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Question What is the strength of the general form?

[Ananyan-Hochster, 2020; Erman-Sam-Snowden, 2020; Bik-Draisma-Eggermont, 2019; Kazhdan-Ziegler, 2018; Derksen-Eggermont-Snowden, 2017; ...]

STATE-OF-THE-ART

Szabò, 1996

Results about dimension of complete intersections contained in general hypersurfaces.

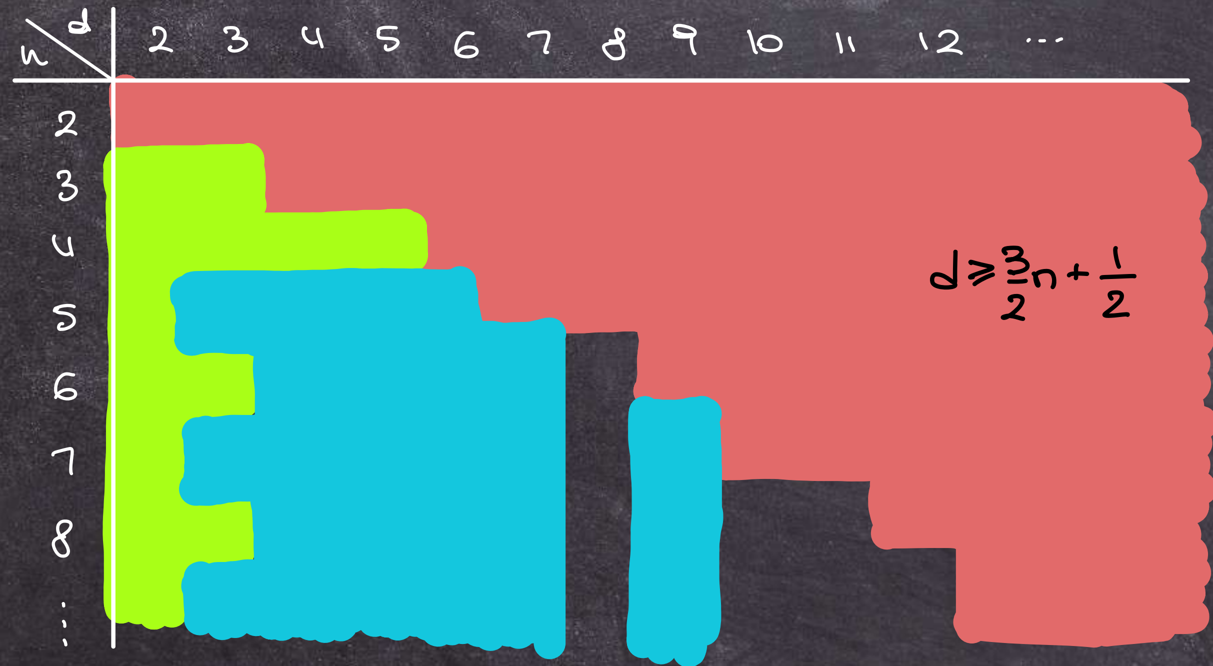
STATE-OF-THE-ART

Szabò, 1996

Carlini - Chiantini - Geramita, 2008

Approach via dimensions of secant varieties

STATE-OF-THE-ART



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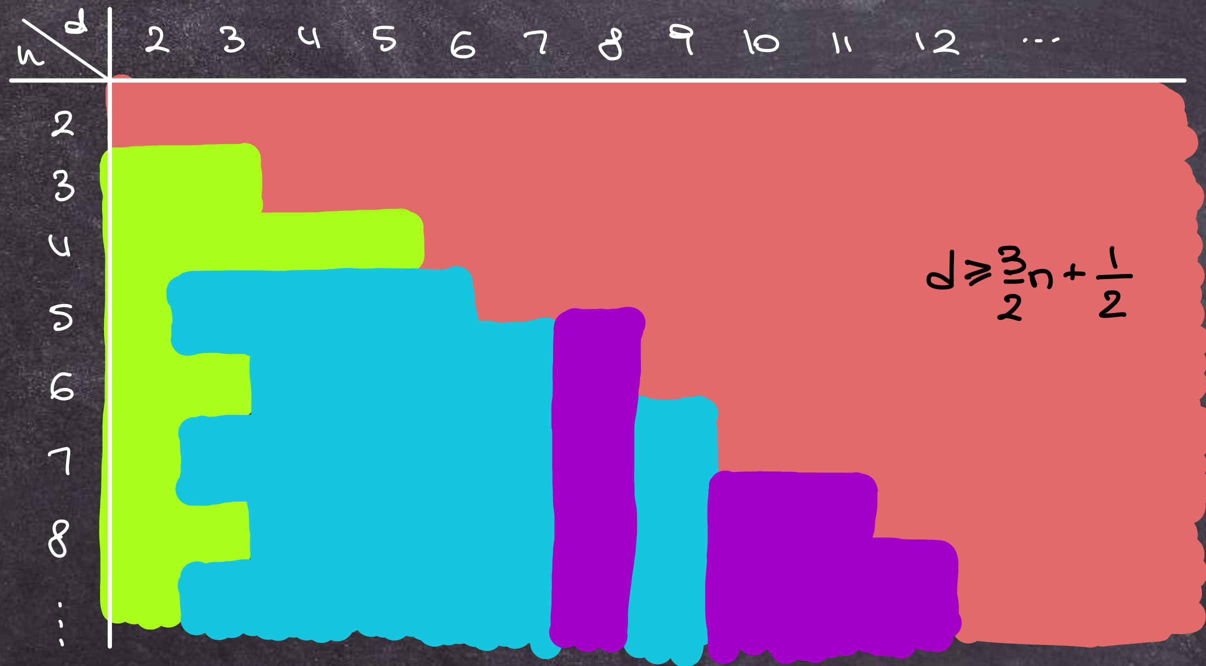

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Bik-Oneto, 2020

Bik-Ballico-Oneto-Ventura, 2021

THE ANSWER

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A slice decomposition of a homogeneous polynomial $f \in S_d$ is

$$f = \ell_1 h_1 + \ell_2 h_2 + \dots + \ell_r h_r \qquad 1 = \deg(\ell_i)$$

The slice rank of $f \in S_d$ is the smallest length of a strength decomposition.

Remark For any f, $\operatorname{str}(f) \leq \operatorname{sl.rk}(f)$.

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Remark The difference $\mathrm{sl.rk}(f) - \mathrm{str}(f)$ can be arbitrarly large

- $f_{n,d} = x_0^d + \ldots + x_n^d$, then $\mathrm{sl.rk}(f_{n,d}) = \lceil \frac{n+1}{2} \rceil$
- $\mathrm{sl.rk}(f_{n,a} \cdot f_{n,b}) = \lceil \frac{n+1}{2} \rceil$, $\mathrm{str}(f_{n,a} \cdot f_{n,b}) = 1$

THE ANSWER

A **strength decomposition** of a homogeneous polynomial $f \in S_d$ is

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THEOREM [Bik-Ballico-Oneto-Ventura, 2021]

For a general form, strength and slice rank coincide.

GENERAL SLICE RANK

A slice decomposition of a homogeneous polynomial $f \in S_d$ is

$$f = \ell_1 h_1 + \ell_2 h_2 + \dots + \ell_r h_r \qquad 1 = \deg(\ell_i)$$

The slice rank of $f \in S_d$ is the smallest length of a strength decomposition.

⟶ The slice rank of a form $f \in S_d$ corresponds to the minimal codimension of a linear space in the hypersurface $\{f = 0\}$.

THEOREM (Classical, see e.g. Harris 1992)
The slice rank of a general form $f \in S_d$ is

$$\min \left\{ r \in \mathbb{Z}_{\geq 0} : r(n+1-r) \geq \binom{d+n-r}{d} \right\}$$

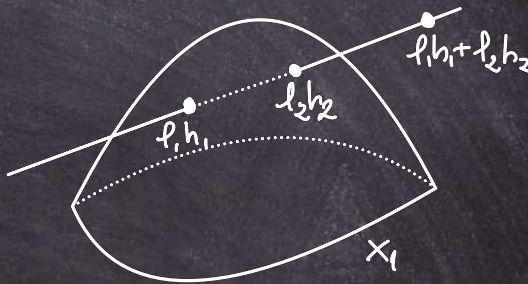
SECANT VARIETIES AND TERRACINI'S LEMMA

Consider the variety of degree-d forms with a linear factor

$$X_1 = \{ [\ell h] : \ell \in S_1 , h \in S_{d-1} \} \subseteq \mathbb{P}(S_d)$$

and its secant varieties

$$\sigma_r(X_1) = \overline{\{ [f] : \mathrm{sl.rk}(f) \leq r \}} \subseteq \mathbb{P}(S_d)$$



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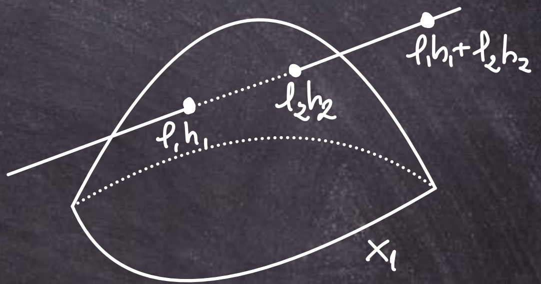

→ The slice rank of a general form in S_d is the minimal r such that

$$\sigma_r(X_1) = \mathbb{P}(S_d)$$

→ The space of forms with bounded slice rank is Zariski-closed

Hence: general slice rank = maximal slice rank

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> **Corollary (Bik-Ballico-Oneto-Ventura, 2021)**
>
> general strength = maximal strength

Indeed: general strength ≤ maximal strength

≤ maximal slice rank = general slice rank

→ The space of forms with bounded slice rank is Zariski-closed

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> **Corollary** (Bik-Ballico-Oneto-Ventura, 2021)
>
> general strength = maximal strength

Indeed: general strength ≤ maximal strength
≤ maximal slice rank = general slice rank

... even if ...

the set of forms with bounded strength is not always Zariski closed.

[Bik-Ballico-Oneto-Ventura, 2020]

$$\{f : \mathrm{str}(f) \leq 3\} \subseteq \mathbb{P}S_4 \text{ is not closed for } n \gg 0.$$

SECANT VARIETIES AND TERRACINI'S LEMMA

Consider the variety of degree-d forms with a linear factor

$$X_1 = \{ [\ell h] : \ell \in S_1, h \in S_{d-1} \} \subseteq \mathbb{P}(S_d)$$

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$$\sigma_r(X_1) = \{ [f] : \mathrm{sl.rk}(f) \leq r \} \subseteq \mathbb{P}(S_d)$$

Terracini's Lemma (1911)

Let $p_1, \ldots, p_r \in X \subseteq \mathbb{P}^N$ general and $p \in \langle p_1, \ldots, p_r \rangle$ general.

Then

$$T_p \sigma_r(X) = \langle T_{p_1}(X), \ldots, T_{p_r}(X) \rangle$$

SECANT VARIETIES AND TERRACINI'S LEMMA

→ $\left.\frac{d}{dt}\right|_{t=0} (\ell + t\ell')(h + th') = \ell' h + \ell h'$, i.e., $T_{[\ell h]} X_1 = \mathbb{P}(\ell, h)_d$.

→ By Terracini's Lemma, $\dim \sigma_r(X_1) = \dim (\ell_1, h_1, \dots, \ell_r, h_r)_d - 1$.

Hence, $\operatorname{codim} \sigma_r X_1 = HF\left(d; \frac{\mathbb{C}[x_0, \dots, x_n]}{(\ell_1, h_1, \dots, \ell_r, h_r)}\right)$

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$= HF\left(d ; \frac{\mathbb{C}[x_0, \ldots, x_{n-r}]}{(h_1, \ldots, h_r)}\right) = \max\left\{0, \binom{d+n-r}{d} - r(n-r+1)\right\}$

[Hochster-Laksov, 1987]

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[Hochster-Laksov, 1987]

THEOREM (Classical, see e.g. Eisenbud-Harris 2016)

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SECANT VARIETIES AND TERRACINI'S LEMMA

Consider the variety of degree-d reducible forms

$$X_{red} = \{ f \ : \ f \text{ is reducible} \} = \bigcup_{j=1}^{\lfloor d/2 \rfloor} X_j \subseteq \mathbb{P} S_d$$

where $X_j = \{ [gh] : g \in S_j \}$.

The r-secant variety is

$$\sigma_r(X_{red}) = \bigcup_{j_1, \dots, j_r} \mathrm{Join}(X_{j_1}, \dots, X_{j_r})$$

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→ $\frac{d}{dt}\Big|_{t=0} (g+tg')(h+th') = g'h + gh'$, i.e., $T_{[gh]} X_j = \mathbb{P}(g,h)_d$.

→ By Terracini's Lemma,

$$\dim J(X_{j_1}, \dots, X_{j_r}) = \dim (g_1, h_1, \cdots, g_r, h_r)_d - 1 .$$

The goal is to study the dimension of $(f_1, h_1, \dots, f_r, h_r)_d$

where $\deg(f_i) + \deg(h_i) = d$, f_i's and h_i's are general.

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Fröberg's Conjecture (1985)

Let $f_1, \dots, f_r$ be general forms with $\deg(f_i) = d_i$ in $n+1$ variables

$$HS\left(S/(f_1, \dots, f_r)\right) = \sum_{d \geq 0} HF\left(d; S/(f_1, \dots, f_r)\right) t^d$$

$$= \left[\frac{\prod_{i=1} (1 - t^{d_i})}{(1-t)^{n+1}} \right]_+$$

where $[-]_+$ is the truncation at the first non-positive coefficient.

$n = 1$: Fröberg 1985 $\qquad$ $r = n+2$: Stanley 1978

$n = 2$: Anick 1986 $\qquad$ asympthotic results : Nenashev 2017

THE MAIN THEOREM

[Catalisano - Geramita - Gimigliano - Harbourne - Migliore - Nagel - Shin, 2019]

If $2r \leq n+1$, then $\dim \sigma_r(X_{red}) = \dim \sigma_r(X_1)$.

They conjectured that this was true for any r, d, n.

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Theorem [Bik-Oneto, 2020 + Bik-Ballico-Oneto-Ventura, 2021]

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Theorem [Bik-Oneto, 2020 + Bik-Ballico-Oneto-Ventura, 2021]

- $\dim \sigma_r(X_{red}) = \dim \sigma_r(X_1)$
- If $\sigma_r(X_{red}) \neq \mathbb{P}S_d$, then for any (n,d,r) $\sigma_r(X_1)$ is the UNIQUE component of maximal dimension except for $(n,d,r) = (3,4,2)$ where

$$\operatorname{codim} \sigma_2(X_1) = \operatorname{codim} J(X_1, X_2) = \operatorname{codim} \sigma_2(X_2) = 1$$

Corollary Strength and slice rank are generically equal

IDEA OF THE PROOF

Assume $r <$ general slice rank.

1) For any $j_1, \dots, j_r \in \{1, \dots, \lfloor d/2 \rfloor\}$, we find an upper bound

$$\dim J(X_{j_1}, \dots, X_{j_r}) \leq c(j_1, \dots, j_r; n, d, r);$$

2) We prove that

$$c(j_1, \dots, j_r; n, d, r) \leq \dim \sigma_r(X_1).$$

IDEA OF THE PROOF

Assume $r <$ general slice rank $\leq n$

1) For any $j_1, \dots, j_r \in \{1, \dots, \lfloor d/2 \rfloor\}$, we find an upper bound

$$\dim J(X_{j_1}, \dots, X_{j_r}) \leq c(j_1, \dots, j_r ; n, d, r) ;$$

In [Bik-Oneto] :

$$I = (g_1, h_1, \dots, g_r, h_r)$$
$$I' = (g_1, h_1, \dots, g_r)$$

$$\left(\frac{S}{I'}\right)_{d-\deg(h_r)} \xrightarrow{\cdot h_r} \left(\frac{S}{I'}\right)_d \longrightarrow \left(\frac{S}{I}\right)_d \longrightarrow 0$$

IDEA OF THE PROOF

Let

$$J^{\circ}_{j_1\dots j_r} = \left\{ f \in S_d \ : \ \begin{array}{l} f = \sum_{i=1}^{r} g_i h_i , \quad \deg(g_i) = j_i \\ (g_1,\dots,g_r) \text{ is a CI} \end{array} \right\}$$

Lemma 1 $J^{\circ}_{j_1\dots j_r}$ is Zariski-dense in $J(X_{j_1},\dots,X_{j_r})$

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Lemma 1 $J^{\circ}_{j_1\dots j_r}$ is Zariski-dense in $J(X_{j_1},\dots,X_{j_r})$

Lemma 2 $\dim J(X_{j_1},\dots,X_{j_r}) \le \dim CI_n(j_1,\dots,j_r) + N$

where:

- $CI_n(j_1,\dots,j_r)$ = set of complete intersections in $\mathbb{P}^n$ defined by polynomials of degrees $j_1,\dots,j_r$
- $N = \dim (g_1,\dots,g_r)_d - 1$ where the g_i's form a CI and $\deg(g_i) = j_i$

IDEA OF THE PROOF

Lemma 2 $\dim \mathcal{J}(X_{j_1,\dots,X_{j_r}}) \leq \dim CI_n(j_1,\dots,j_r) + N$

where :
- $CI_n(j_1,\dots,j_r)$ = set of complete intersections in $\mathbb{P}^n$ defined by polynomials of degrees $j_1,\dots,j_r$
- $N = \dim (g_1,\dots,g_r)_d - 1$ where the g_i's form a CI and $\deg(g_i) = j_i$

Let $E = \{ (Y, [f]) \in CI_n(j_1,\dots,j_r) \times \mathbb{P}S_d : f \in (I_Y)_d \}$

The projection on the second factor gives a surjective map

$$E \twoheadrightarrow \mathcal{J}^\circ_{j_1,\dots,j_r}.$$

IDEA OF THE PROOF

Lemma 3

$$\dim CI_n(j_1,\dots,j_r) = \sum_{i=1}^{r} \operatorname{coeff}_{j_i}\left(\frac{\prod_{i=1}^{r}(1-t^{j_i})}{(1-t)^{n+1}}\right)$$

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$P(j_1,\dots,j_r)$: Hilbert polynomial of $Y \in CI_n(j_1,\dots,j_r)$

$CI_n(j_1,\dots,j_r)$ is parametrized by a Zariski-open subset of $\mathrm{Hilb}_{P(j_1\cdots j_r)}(\mathbb{P}^n)$

and the latter is smooth at $[Y] \in CI_n(j_1,\dots,j_r)$

with $T_{[Y]}\mathrm{Hilb}_{P(j_1\cdots j_r)}(\mathbb{P}^n) = H^0(N_{Y/\mathbb{P}^n})$ [Sernesi 2006]

Moreover, $N_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^{r} \mathcal{O}_Y(j_i)$

IDEA OF THE PROOF

Theorem Let $r<n$, $j_1,\dots,j_r \leq d/2$ and let $\ell := \#\{i : j_i = d/2\}$

Then,

$$\dim J(X_{j_1} \dots X_{j_r}) \leq \binom{n+d}{d} - \operatorname{coeff}_d\left(\frac{\prod_{i=1}^{r}(1-t^{j_i})(1-t^{d-j_i})}{(1-t)^{n+1}}\right) + \binom{\ell}{2} - 1$$

$$=: c(j_1,\dots,j_r)$$

... then we need a careful numerical analysis to deduce

$$c(j_1,\dots,j_r) \leq c(j_1,\dots,j_r - 1) \leq \dots \leq c(1,\dots,1) = \dim \sigma_r X_1$$

THANK YOU !

Bik, Oneto – "On the strength of general polynomials"
Lin. and Multilin. Algebra, 2021

Bik, Ballico, Oneto, Ventura – "The set of forms with bounded strength is not closed", arXiv:2012.01237

Bik, Ballico, Oneto, Ventura – "Strength and slice rank are generically equal", arXiv:2102.11549