Variants of Comon's problem via simultaneous ranks

Alessandro Oneto (Otto-von-Guericke-Universität Magdeburg)
based on a joint work with F. Gesmundo and E. Ventura

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OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG
Alexander von Humboldt Stiftung/Foundation

Introduction: Comon's question

Additive decompositions of tensors

Let $V_1, \ldots, V_k$ be vector spaces and $\mathrm{Sym}^d(V)$ be the space of symmetric tensors in $V^{\otimes d}$.

Structure	Decomposition	Rank
tensors $t \in V_1 \otimes \ldots \otimes V_k$	$t = \sum_{i=1}^r v_{i,1} \otimes \ldots \otimes v_{i,k}$	$R_{(1,\ldots,1)}(t)$
partially symmetric tensors $t \in \mathrm{Sym}^{d_1} V_1 \otimes \ldots \otimes \mathrm{Sym}^{d_k} V_k$	$t = \sum_{i=1}^r v_{i,1}^{\otimes d_1} \otimes \ldots \otimes v_{i,k}^{\otimes d_k}$	$R_{(d_1,\ldots,d_k)}(t)$
symmetric tensors $t \in \mathrm{Sym}^d V$	$t = \sum_{i=1}^r v^{\otimes d}$	$R_d(t)$
$f \in \Bbbk[x_0, \ldots, x_n]_d$	$f = \sum_{i=1}^r \ell_i^d$	$R_d(f)$

Comon's question I

Let $t \in \mathrm{Sym}^d V$ be a symmetric tensor. Then, for any partition $(d_1, \ldots, d_k) \vdash d$,

$$t \in \mathrm{Sym}^d V \subset \mathrm{Sym}^{d_1} V \otimes \ldots \otimes \mathrm{Sym}^{d_k} V \subset V^{\otimes d}.$$

Example

Say $\dim V = 3$, with basis $\{x, y, z\}$.

$\mathrm{Sym}^3 V$	$t = xyz$
$\mathrm{Sym}^2 V \otimes \mathrm{Sym}^1 V$	$t = \frac{1}{3}(xy \otimes z + xz \otimes y + yz \otimes x)$
$V \otimes V \otimes V$	$t = \frac{1}{6}(x \otimes y \otimes z + x \otimes z \otimes y + y \otimes x \otimes z + \ldots + z \otimes y \otimes x)$

Comon's question II

Given $t \in \mathrm{Sym}^d V \subset V^{\otimes d}$, clearly, $R_{(1,\ldots,1)}(t) \leq R_d(t)$. Indeed,

$$t = \sum_{i=1}^r \ell_i^d \quad \Longrightarrow \quad t = \sum_{i=1}^r \ell_i \otimes \ldots \otimes \ell_i.$$

QUESTION (Comon, 2008)

Is it true that $R_d(t) = R_{(1,\ldots,1)}(t)$?

- [Comon-Golub-Li-Mourrain, 2008]
- Binary forms: Yes [Zhang-Huang-Qi, 2016];
- Ternary cubics: Yes [Friedland, 2016];
- Quaternary cubics: Yes [Seigal, 2018];
- In general: NO! Counterexample for $d = 3$ and $\dim V = 800$ [Shitov, 2018].

Comon's question III

Given $t \in \mathrm{Sym}^d V \subset \mathrm{Sym}^{d_1} V \otimes \ldots \otimes \mathrm{Sym}^{d_k} V$, clearly, $R_{(d_1,\ldots,d_k)}(t) \leq R_d(t)$. Indeed,

$$t = \sum_{i=1}^r \ell_i^d \quad \Longrightarrow \quad t = \sum_{i=1}^r \ell_i^{\otimes d_1} \otimes \ldots \otimes \ell_i^{\otimes d_k}.$$

QUESTION

For a partition $(1,\ldots,1) \neq (d_1,\ldots,d_k) \vdash d$,

is it true that $R_{(d_1,\ldots,d_k)}(t) = R_d(t)$?

Comon's question IV

If $(d'_1, \ldots, d'_h)$ is a "refinement" of the partition $(d_1, \ldots, d_k)$, then $R_{(d'_1,\ldots,d'_h)}(t) \leq R_{(d_1,\ldots,d_k)}(t)$.

Hence, we ask: In the chain

$$R_{(1,\ldots,1)}(t) \leq \ldots \leq R_{(d_1,\ldots,d_k)}(t) \leq \ldots \leq R_d(t),$$

where can we have strict inequalities?

Two directions:

- find new counterexamples where the chain breaks earlier;
- get positive answers to partially symmetric versions of Comon's question for new families of polynomials.

Simultaneous decompositions and gradient ranks

Simultaneous rank I

Given a set of tensors $\mathcal{T} = \{t_1, \ldots, t_s\} \subset V_1 \otimes \ldots \otimes V_k$, what is the minimal r needed to simultaneously decompose all tensors in $\mathcal{T}$ as

$$t_i = \sum_{j=1}^{r} \lambda_{i,j} v_{j,1} \otimes \ldots \otimes v_{j,k}, \qquad \text{for all } i = 1, \ldots, s?$$

This is the tensorial version of the problem of simultaneous diagonalization of matrices

Simultaneous rank II

In the symmetric case, given a set of degree-d forms $\mathcal{F} = \{f_1, \ldots, f_s\}$, what is the minimal r needed to simultaneously decompose all elements of $\mathcal{F}$ as

$$R(\mathcal{F}) = \min \left\{ r \mid \exists \ell_1, \ldots, \ell_r \quad \text{s.t.} \quad f_i = \sum_{j=1}^{r} \lambda_{i,j} \ell_j^d, \qquad \text{for all } i = 1, \ldots, s \right\}?$$

Such minimal r is called the **simultaneous rank** of $\mathcal{F}$; denoted $R(\mathcal{F})$

Simultaneous rank III

As regards sets of "general" forms:

- If $\mathcal{F}$ is a pair of general quadrics in n variables, $R(\mathcal{F}) = n$ [Weierstrass, ∼1860]
- Geometric approach: Grassmann defectivity [Terracini, 1915; Fontanari, 2002]
- Identifiability questions (see e.g. [Angelini-Galuppi-Mella-Ottaviani, 2016])

As regards "specific" sets of forms:

- Special sets of monomials [Carlini-Ventura, 2017]

Gradient rank I

Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a homogeneous polynomial. Then, we consider

$$\nabla(f) = \left\{ \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \right\}.$$

We define the **gradient rank** of f as the simultaneous rank of $\nabla(f)$, i.e.,

$$R_{\nabla}(f) := R(\nabla(f)).$$

Gradient rank II

Clearly, if $f = \sum_{i=1}^{r} \ell_i^d$, then $\frac{\partial f}{\partial x_i} = d \sum_{i=1}^{r} \lambda_i \ell_i^{d-1}$; hence, $R(f) \geq R_\nabla(f)$.

QUESTION

Let $f \in \mathbb{k}[x_0, \ldots, x_n]$ be a homogeneous polynomial. Is it true that

$$R(f) = R_\nabla(f)?$$

REMARK

For general tensors, the analogous question is true. I.e., if $t \in V_1 \otimes \ldots \otimes V_k$, then the simultaneous rank of $\left\{ t(w_i^*) \in V_1 \otimes \ldots \widehat{V_i} \ldots \otimes V_k \mid w_i^* \in V_i^* \right\}$ is equal to the rank of t.

Gradient rank III

LEMMA (Teitler, 2014)

Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then,

$$R_{(1,d-1)}(f) = R_{\nabla}(f).$$

Idea: The simultaneous rank of $\mathcal{F} = \{f_1, \ldots, f_k\} \subset \mathrm{Sym}^d V$ equals the partially symmetric rank of $\sum_{i=1}^k a_i \otimes f_i$ in $A \otimes \mathrm{Sym}^d V$. Indeed,

$$f_i = \sum_{j=1}^{r} c_{i,j} \ell_j^{d-1}, \quad \text{for all } i = 1, \ldots, k \quad \Longleftrightarrow \quad \sum_{i=1}^{k} a_i \otimes f_i = \sum_{j=1}^{r} (c_{1,j} a_1 + \ldots + c_{k,j} a_k) \otimes \ell_j^{d-1}.$$

Here, we look at $f = \sum_{i=0}^n x_i \otimes \frac{\partial f}{\partial x_i}$.

k-th Gradient rank I

Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a homogeneous polynomial. Then, we consider

$$\nabla^k(f) = \left\{ \frac{\partial^k}{\partial x_0^{a_0} \cdots \partial x_n^{a_n}}(f) \mid a_0 + \ldots + a_n = k \right\}.$$

We define the **k-th gradient rank** of f as the simultaneous rank of $\nabla^k(f)$, i.e.,

$$R_{\nabla^k}(f) := R(\nabla^k(f)).$$

k-th Gradient rank II

LEMMA

Let $f \in \Bbbk[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d.
Let $(d_1, \ldots, d_m) \vdash d$, with $d_m = d - k$, then

$$R_{(d_1,\ldots,d_{m-1},d-k)}(f) \geq R_{\nabla^k}(f).$$

We cannot hope for equality in general: pick the 2nd gradient of a cubic of rank $> n+1$.

However, we can look at the chain

$$R_d(f) \geq R_{(d_1,\ldots,d_{m-1},d-k)}(f) \geq R_{\nabla^k}(f).$$

Cactus versions

Geometric interpretation of ranks

Given a projective variety $X \subset \mathbb{P}^N$, the **X-rank** of a point $p \in \mathbb{P}^N$ is

$$R_X(p) := \min\{r \mid \exists q_1, \ldots, q_r \in X \quad \text{s.t.} \quad p \in \langle q_1, \ldots, q_r \rangle\};$$

the **simultaneous X-rank** of a set of points $\mathcal{P} = \{p_1, \ldots, p_s\} \subset \mathbb{P}^N$ is

$$R_X(\mathcal{P}) := \min\{r \mid \exists q_1, \ldots, q_r \in X \quad \text{s.t.} \quad \langle p_1, \ldots, p_s \rangle \subset \langle q_1, \ldots, q_r \rangle\}.$$

In particular,

tensor rank / partially symmetric rank / symmetric rank
correspond to
Segre / Segre-Veronese / Veronese varieties

Cactus versions of ranks

Given a projective variety $X \subset \mathbb{P}^N$, the **cactus X-rank** of a point $p \in \mathbb{P}^N$ is

$$cR_X(p) := \min\{r \mid \exists \mathbb{X} \subset X, \ \deg(\mathbb{X}) = r, \quad \text{s.t.} \quad p \in \langle \mathbb{X} \rangle\};$$

the **simultaneous cactus X-rank** of a set of points $\mathcal{P} = \{p_1, \ldots, p_s\} \subset \mathbb{P}^N$ is

$$cR_X(\mathcal{P}) := \min\{r \mid \exists \mathbb{X} \subset X, \deg(\mathbb{X}) = r \quad \text{s.t.} \quad \langle p_1, \ldots, p_s \rangle \subset \langle \mathbb{X} \rangle\}.$$

Cactus version of Comon's question

Let $f \in \mathrm{Sym}^d V$. Then, we still have a chain of inequalitites

$$cR_d(f) \geq \ldots \geq cR_{\underline{d}}(f) \geq \ldots \geq cR_{(1,\ldots,1)}(f).$$

QUESTION

When do we have equalities/inequalities?

Cactus gradient ranks

Let $f \in \mathrm{Sym}^d V$. Then, the **k-th gradient cactus rank** is $cR_{\nabla^k}(f) := cR(\nabla^k(f))$.

LEMMA

Let $\underline{d} = (d_1, \ldots, d_{m-1}, d-k) \vdash d$. Then,

$$cR_{\underline{d}}(f) \geq cR_{\nabla^k}(f).$$

REMARK

It is no longer true that the symultaneous cactus rank of a family $\mathcal{F} = \{f_1, \ldots, f_k\}$ equals the cactus rank of $\sum_{i=1}^k x_i \otimes f_i$. Counterexample given by $\mathcal{F} = \{x_0x_1^2, x_0x_2^2\}$. Indeed, $cR(\mathcal{F}) = 3$ while $cR_{(1,2)}(x_0 \otimes x_0x_1^2 + x_1 \otimes x_0x_2^2) \geq 4$.

Results

Apolarity

The idea to compute gradient ranks (cactus ranks, resp.) of homogeneous polynomials is classical and exploits Apolarity Theory. Indeed,

$R_{\nabla^k}(f)$ is the minimal cardinality of a set of reduced points $\mathbb{X} \subset \mathbb{P}^n$ s.t.
($cR_{\nabla^k}(f)$ is the minimal degree of a 0-dim. scheme $\mathbb{X} \subset \mathbb{P}^n$, resp.)

$$I_{\mathbb{X}} \subset \bigcap_{a_0+\ldots+a_n=k} \left(\frac{\partial^k f}{\partial x_0^{a_0} \cdots \partial x_n^{a_n}} \right)^{\perp} = f^{\perp} + \mathrm{Sym}^{d-k+1} V^*.$$

Binary forms I

THEOREM

Let $f \in \mathrm{Sym}^d V$, with $\dim V = 2$. Then, for any $k < d$,

$$R_{\nabla^k}(f) = \min\{R_d(f), d-k+1\} \quad \text{and} \quad cR_{\nabla^k}(f) = \min\{cR_d(f), d-k+1\}.$$

Consequently, for any $\underline{d} \vdash d$ with $d_m = d-k$, we have:

- if $R_d(f) \leq d-k+1$, then $R_d(f) = R_{\underline{d}}(f) = R_{\nabla^k}(f)$;
- if $cR_d(f) \leq d-k+1$, then $cR_d(f) = cR_{\underline{d}}(f) = cR_{\nabla^k}(f)$.

Binary forms II

From the proof, we also deduce more:

- if $R_d(f) < d - k + 1$, then the sets of points minimally spanning f are the same as the ones minimally spanning $\nabla^k(f)$ (analogously, for cactus);
- if $R_d(f) \geq d - k + 1$, then any set of $d - k + 1$ points span $\nabla^k(f)$;
- if $R_d(f) = d - k + 1$, then the rank and the k-th gradient rank are the same, but we can find minimal sets of points minimally spanning $\nabla^k(f)$ which do not minimally span f. E.g., for $k = 1$
  $x_0x_1^{d-1}$ cannot be minimally spanned by d points involving x_1^d, but $\{x_0x_1^{d-2}, x_1^{d-1}\}$ can be minimally spanned by d points involving x_1^{d-1}.

Other forms I

THEOREM

Let $f \in \mathrm{Sym}^3 V$, with $\dim V = 3$ or 4. Then,

$$R_3(f) = R_{(1,2)}(f) = R_\nabla(f)$$
$$cR_3(f) = cR_{(1,2)}(f) = cR_\nabla(f).$$

REMARK

Let $f = x_0(x_0x_1 + x_2^2)$ – ternary cubic of maximal rank $R(f) = 5$.

There are no minimal decompositions of f involving x_0^3, but there is a minimal decomposition of ∇f involving x_0^2.

Other forms II

THEOREM

Let $m = x_0^{a_0} \cdots x_n^{a_n}$. Let $k \leq \min\{a_i\}$ and $d = \deg(m)$. For $\underline{d} = (d_1, \ldots, d_{m-1}, d-k) \vdash d$

$$R_d(m) = R_{\underline{d}}(m) = R_{\nabla^k}(m);$$

for $k = 1$,

$$cR_d(m) = cR_{(1,d-1)}(m) = R_{\nabla}(m).$$

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Conclusions

$$\text{Is it } R(f) = R_{(1,d-1)}(f) = R_{\nabla}(f)?$$

$$\text{Is it } cR(f) = R_{(1,d-1)}(f) = cR_{\nabla}(f)?$$

Yes, for:
binary forms, ternary and quaternary cubics, monomials,
elementary symmetric polynomials of odd degree.

Gesmundo, F., Oneto, A., Ventura, E. – Partially symmetric variants of Comon's problem via symultaneous rank – arXiv:1810.07679

THANK YOU!