A new question on planar polynomial interpolation and line arrangements

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(joint work with M. Di Marca[1] and G. Malara[2])

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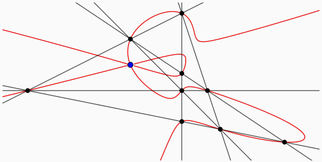


[1] DIMA - Università degli Studi di Genova, Italy [2] Pedagogical University of Cracow, Poland

Introduction

Polynomial interpolation: simple points

Polynomial Interpolation Problem

Given a set of points $\mathbb{X} = \{P_1, \ldots, P_d\}$ in complex projective plane $\mathbb{P}^2$,
how many curves of degree j pass through $\mathbb{X}$?

e.g., through 2 distinct points there is a unique line

through 5 general points there is a unique conic

Polynomial interpolation: simple points

Let $S = \mathbb{C}[x_0, x_1, x_2] = \bigoplus_{j\geq 0} S_j$, standard graded polynomial ring.

$S_j :=$ $\mathbb{C}$-vector space of homogeneous polynomials of degree j

Hilbert function

Let $I = \bigoplus_{j\geq 0} I_j$ be a homogeneous ideal. The **Hilbert function** of S/I in degree j is

$$\mathrm{HF}_{S/I}(j) := \dim_{\mathbb{C}} S_j/I_j = \dim_{\mathbb{C}} S_j - \dim_{\mathbb{C}} I_j.$$

Let $\mathbb{X} = \{P_1, \ldots, P_d\} \subset \mathbb{P}^2$, then $I(\mathbb{X}) = \wp_1 \cap \ldots \cap \wp_d = \bigoplus_{j\geq 0} I(\mathbb{X})_j \subset S$.

$$P = (p_0 : p_1 : p_2) \leftrightsquigarrow \wp = (p_1x_0 - p_0x_1, p_2x_0 - p_0x_2)$$

The **Hilbert function** of $\mathbb{X}$ is the Hilbert function of $S/I(\mathbb{X})$.

Polynomial interpolation: simple points

Polynomial Interpolation Problem

Given a set of points $\mathbb{X} = \{P_1, \ldots, P_d\}$ in complex projective plane $\mathbb{P}^2$, what is the Hilbert function of $\mathbb{X}$ in degree j?

Obviously, the answer depends on the position of the points.

[Geramita-Orecchia, 1981] If the points are in **general position**,

$$\mathrm{HF}_{\mathbb{X}}(j) = \min\left\{\binom{j+2}{2}, d\right\}.$$

Proof. If $\{m_1, \ldots, m_{\binom{j+2}{2}}\}$ is the standard monomial basis for S_j, then $\mathrm{HF}_{\mathbb{X}}(j) = \mathrm{rk}\,(m_a(P_b))_{a,b}$.

Polynomial interpolation: fat points

Fat points

A **fat point** of **multiplicity** m and **support** at P is the 0-dim scheme given by $\wp^m$. We denote it by mP.

A **scheme of fat points** is a union of fat points, i.e., $\mathbb{X} = m_1P_1 + \ldots + m_dP_d$ defined by $I(\mathbb{X}) = \wp_1^{m_1} \cap \ldots \cap \wp_d^{m_d}$.

Remark. $f \in \wp^m$ if and only if $D(f)|_P = 0$, for any $D \in \mathbb{C}[\partial_0, \partial_1, \partial_2]_{\leq m-1}$.

Polynomial interpolation: fat points

Polynomial Interpolation Problem

Let $\mathbb{X} = m_1P_1 + \ldots + m_dP_d$ be a scheme of fat points in $\mathbb{P}^2$,

what is the Hilbert function of $\mathbb{X}$ in degree j?

this is equivalent to asking

Polynomial interpolation: fat points

Polynomial Interpolation Problem

Let $\mathbb{X} = m_1P_1 + \ldots + m_dP_d$ be a scheme of fat points in $\mathbb{P}^2$,

what is the Hilbert function of $\mathbb{X}$ in degree j?

this is equivalent to asking

Given a set of points $\{P_1, \ldots, P_d\}$ and positive integers $m_1, \ldots, m_d$,

how many curves of degree j are singular at P_i of order m_i, for $i = 1, \ldots, d$?

Polynomial interpolation: fat points

Remark. $f \in \wp^m$ if and only if $D(f)|_P = 0$, for any $D \in \mathbb{C}[\partial_0, \partial_1, \partial_2]_{\leq m-1}$.

Therefore, a fat point of multiplicity m in $\mathbb{P}^2$ imposes $\binom{m+1}{2}$ linear equations.

If we assume $\mathbb{X} = m_1 P_1 + \ldots + m_d P_d$ to have general support, then, the **expected Hilbert function** of $\mathbb{X}$ in degree j is

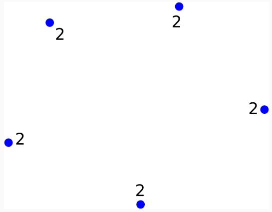
$$exp.\mathrm{HF}_{\mathbb{X}}(j) = \min \left\{ \binom{j+2}{2}, \sum_{i=1}^{d} \binom{m_i + 1}{n} \right\}.$$

Polynomial interpolation: fat points

Example 1. Let $\mathbb{X} = 2P_1 + \ldots + 2P_5 \subset \mathbb{P}^2$, with general support.

We expect to have no quartics through $\mathbb{X}$.

$$exp.\dim I(\mathbb{X})_4 = \binom{4+2}{2} - 5 \cdot 3 = 0.$$



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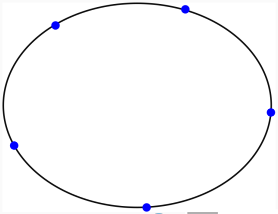

Polynomial interpolation: fat points

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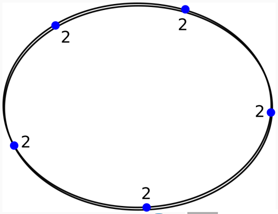
$$exp. \dim I(\mathbb{X})_4 = \binom{4+2}{2} - 5 \cdot 3 = 0.$$

There is a unique conic C passing through the points. Hence, $2C \in I(\mathbb{X})_4$. By Bézout's Theorem,

$$\dim I(\mathbb{X})_4 = 1 > 0 = exp. \dim I(\mathbb{X})_4.$$

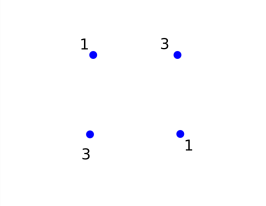


Polynomial interpolation: fat points

Example 2. Let $\mathbb{X} = 3P_1 + 3P_2 + P_3 + P_4 \subset \mathbb{P}^2$, with general support.

We expect to have no quartics through $\mathbb{X}$.

$$exp.\dim I(\mathbb{X})_4 = \binom{4+2}{2} - 2 \cdot 6 - 2 = 1.$$



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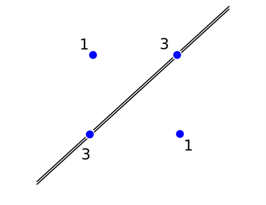
The line $L = \overline{P_1 P_2}$ is a fixed component.



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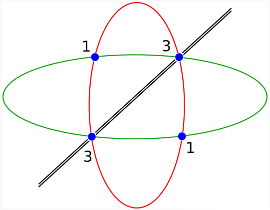
$$exp.\dim I(\mathbb{X})_4 = \binom{4+2}{2} - 2\cdot 6 - 2 = 1.$$

The line $L = \overline{P_1P_2}$ is a fixed component.
If $\langle C_1, C_2 \rangle = I(P_1 + P_2 + P_2 + P_4)$, then $C_1L^2, C_2L^2 \in I(\mathbb{X})_4$, and

$$\dim I(\mathbb{X})_4 = 2 > 1 = exp.\dim I(\mathbb{X})_4.$$



SHGH Conjecture

BENIAMINO SEGRE

ALCUNE QUESTIONI SU INSIEMI FINITI DI PUNTI IN GEOMETRIA ALGEBRICA

È con viva commozione che mi accingo a parlare nella mia città natìa, in questa gloriosa Università che mi accolse sedicenne e dove trascorsi il periodo più formativo e determinante dei miei studi. Sono lieto di rivedere qui presenti, e vegeti quasi che da allora non fossero trascorsi ben otto lustri, i miei professori di quegli anni BOGGIO e TOGLIATTI, ed il professore TERRACINI del quale fui poi per qualche tempo assistente.

Il mio pensiero si volge con gratitudine ad essi ed agli altri miei Maestri: CORRADO SEGRE con cui mi addottorai, SOMIGLIANA del quale pure fui assistente, ed inoltre PEANO e FANO, tutti purtroppo scomparsi, ma la cui voce ed i cui insegnamenti ancora mi riecheggiano nel cuore.

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Osserviamo anzitutto che, essendo $\delta \geqslant -1$, *risulta sempre* $\sigma > 0$ *quando si supponga* $d < -1$. Altri esempi di sistemi lineari Σ sovrabbondanti, aventi cioè appunto $\sigma > 0$, vengono offerti da

$$\{10 \mid 4^2, 5^3\}$$

e da

$$\{n \mid k_1, k_2\} \qquad \text{con } k_1, k_2 \leq n,\ k_1 + k_2 \geq n+2,$$

per i quali rispettivamente si ha $d = 0$, $\sigma = 2$ e

$$\sigma = (k_1 + k_2 - n)(k_1 + k_2 - n - 1)/2;$$

e va rilevato al riguardo come ciascuno di questi sistemi risulti dotato di componente fissa multipla (rispettivamente: la conica per i cinque punti base contata 4 volte e la retta per i due punti base contata $k_1 + k_2 - n$ volte). Questi ed altri esempi consimili portano a fare ritenere probabile che

Affinchè un sistema lineare completo Σ di curve piane, dotato di un numero finito di punti base assegnati in posizione generica ed avente dimensione virtuale $d \geqslant -1$, sia **sovrabbondante** *(e quindi effettivo, cioè di dimensione $\delta \geqslant 0$) è* **necessario** *(ma, come risulta da esempi, non sufficiente) ch'esso possegga qualche* **componente fissa multipla**.

SHGH Conjecture

B. Segre 1961

Alcune questioni su insiemi finiti di punti in geometria algebrica (page 72)

"(...) in order that a complete linear system Σ of plane curves, passing through multiple base points in general position with virtual dimension $d \geq -1$ is superabundant (and then effective, i.e., of dimension $\delta \geq 0$), it is necessary (but, from examples, not sufficient) that it has some multiple fixed component."

SHGH Conjecture

A degree d curve C is *negative* for $P_1, \ldots, P_d \in \mathbb{P}^2$ if $(\mathrm{mult}_{P_1}(C))^2 + \ldots + (\mathrm{mult}_{P_d}(C))^2 > d^2$.

[SHGH Conjecture - B. Segre, '61; Harbourne '85; Gimigliano '87; Hirschowitz '89]
If with general support and $f = f_1^{b_1} \cdots f_t^{b_t}$ is the greatest common divisor of $I(\mathbb{X})_j$. Let $\mathcal{N}$ be the set of negative curves for $P_1, \ldots, P_d$. Then,

$$\mathrm{HF}_{\mathbb{X}}(j) = \min \left\{ \binom{j+2}{2}, \sum_{i=1}^{s} \binom{m_i+1}{2} - \sum_{i \,:\, f_i \in \mathcal{N}} \binom{b_i}{2} \right\}.$$

[Castelnuovo, 1891] $d \leq 9$; [Alexander-Hirschowitz, 1995] $m = 2$ (for any $\mathbb{P}^n$);

[Yang, 2007] $m_i \leq 7$; [Ciliberto-Miranda, 1998] $m_i = m \leq 12$; ...and some other special cases

A new planar interpolation problem

A new planar interpolation problem

Problem. [Cook II - Harbourne - Migliore - Nagel, '16]

Let $Z = P_1 + \ldots + P_d \subset \mathbb{P}^n$ be a set of reduced points (non necessary with general support) and consider the linear system $\mathcal{L}_j(Z)$ of curves of degree j passing through Z.

Let $\mathbb{X}$ be a scheme of fat points with general support,

how many conditions does $\mathbb{X}$ impose on $\mathcal{L}_j(Z)$?

$Z = \emptyset$ is the situation of SHGH Conjecture.

A new planar interpolation problem

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$Z = \emptyset$ is the situation of SHGH Conjecture.

We focus on $\mathbb{X} = mQ$, with a Q general point

Unexpected curves

Definition. [Cook II - Harbourne - Migliore - Nagel, '16]

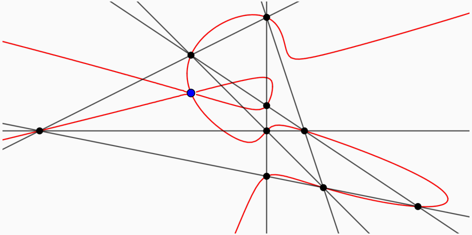
Let $Z = P_1 + \ldots + P_d \subset \mathbb{P}^n$ be a set of reduced points (non necessary with general support).
We say that Z admits an **unexpected curve** of degree $j+1$ if, for general point $Q \in \mathbb{P}^2$

$$\dim_{\mathbb{C}} I(Z + jQ)_{j+1} > \max\left\{0,\ \dim_{\mathbb{C}} I(Z)_{j+1} - \binom{j+1}{2}\right\}$$

Unexpected curves

Questions.

1. *Classify the pairs (Z, j) for which Z admits an unexpected curve of degree $j+1$.*



[Cook II - Harbourne - Migliore - Nagel]

Example of 9 points with an unexpected cubic.

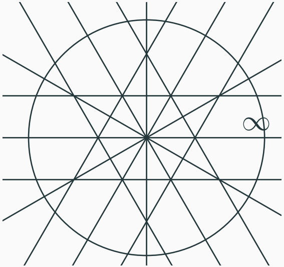
[Farnik - Galuppi - Sodomaco - Trok]

Up to isomorphism, this is the only possible example of 9 points with an unexpected cubic.

Unexpected curves

Questions.

2. *Provide more examples.*



[Di Marca - Malara - Oneto]

Series of examples coming from special line arrangements in $\mathbb{P}^2$.

Line arrangements

Line arrangements

A **line arrangement** is a collection of lines $\mathcal{A} = \{\ell_1, \ldots, L_r\}$

A **singular point** in a line arrangement is a point where at least two lines meet

$$\mathrm{Sing}(\mathcal{A}) = \bigcup_{a \neq b} L_a \cap L_b$$

The **multiplicity** of a singular point is the number of lines of $\mathcal{A}$ meeting at the point

$$m(P) = \#\{L \in \mathcal{A} \mid P \in L\}$$

Dual line arrangement

Given a point $P = (p_0 : p_1 : p_2) \in \mathbb{P}^2$, we define the **dual line**

$$L_P = \{p_0x_0 + p_1x_1 + p_2x_2 = 0\}.$$

Given a set of points $Z = \{P_1, \ldots, P_d\}$, we define a **dual line arrangement**

$$\mathcal{A}_Z = \{L_{P_1}, \ldots, L_{P_d}\}.$$

Splitting type

[Cook II - Harbourne - Migliore - Nagel]

Let $\mathcal{A} = \{L_1, \ldots, L_d\}$ be a line arrangement. Let $f_{\mathcal{A}} = \ell_1 \cdots \ell_d$, where $L_i = \{\ell_i = 0\} \subset \mathbb{P}^2$.

Then we consider the **derivation bundle** $\mathcal{D}_{\mathcal{A}}$ defined as the kernel

$$0 \longrightarrow \mathcal{D}_{\mathcal{A}} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \xrightarrow{\nabla_{\mathcal{A}}} \mathcal{O}_{\mathbb{P}^2}(d-1),$$

where $\nabla_{\mathcal{A}} = [\partial_0 f,\ \partial_1 f,\ \partial_2 f]$.

- $\mathcal{D}_{\mathcal{A}}$ is isomorphic to (a twist of) the *syzygy bundle* of the Jacobian ideal
$$J_{\mathcal{A}} = (\partial_0 f,\ \partial_1 f,\ \partial_2 f) \subset \mathbb{C}[x_0, x_1, x_2];$$
- $\mathcal{D}_{\mathcal{A}}$ is a locally free sheaf of rank 2.

Splitting type

Definition. If $\mathcal{D}_{\mathcal{A}}$ is *free*, i.e., $\mathcal{D}_{\mathcal{A}} = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b)$, we say that

$\mathcal{A}$ is **free** with **splitting type** (a, b).

Remark 1. In this case, the minimal free resolution of the Jacobian ideal is

$$0 \longrightarrow S(-(d-1)-a) \oplus S(-(d-1)-b) \longrightarrow S(-(d-1))^{\oplus 3} \longrightarrow S \longrightarrow S/J_{\mathcal{A}} \longrightarrow 0.$$

Remark 2. If it is not free, we have $\mathcal{D}_{\mathcal{A}}|_L = \mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$, on any line L. This is the *splitting type of $\mathcal{A}$ on L* and it is constant on a open set of the dual projective plane.

In this situation, we define the *splitting type of $\mathcal{A}$* as the splitting type on the general line.

We always have $a + b = d - 1$

Splitting type vs Unexpected curves

Theorem. [Cook II - Harbourne - Migliore - Nagel]

Let $Z \subset \mathbb{P}^2$ be a set of reduced points and let (a, b) be the splitting type of the dual line arrangement $\mathcal{A}_Z$, with $a \leq b$.

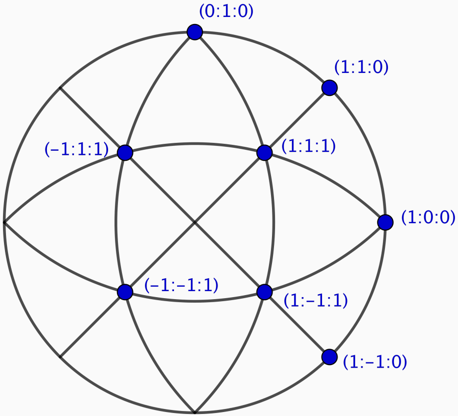
Then, Z admits unexpected curves if and only if

1. $|Z| \geq 2a + 2$;
2. Z does not contain $2a + 2$ collinear points.

In this case, Z admits unexpected curves of degree $j + 1$ if and only if

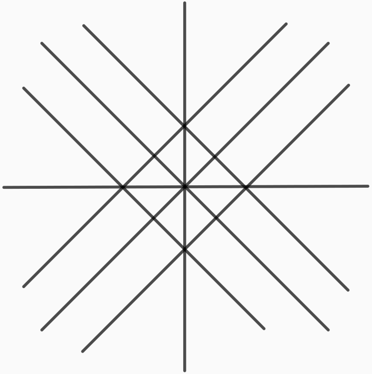
$$a \leq j \leq b - 2.$$

Example [Cook II - Harbourne - Migliore - Nagel]



$$Z = \{(1:0:0),\ (0:1:0),\ (1:-1:0),\ (-1:1:1),\ (1:-1:1),\ (1:1:0),\ (-1:-1:1),\ (1:1:1)\}.$$

Example [Cook II - Harbourne - Migliore - Nagel]



$$f_{\mathcal{A}} = (x)(y)\cdot$$
$$\cdot (x-y)(x-y-z)(x-y+z)\cdot$$
$$\cdot (x+y)(x+y-z)(x+y+z)$$

Example [Cook II - Harbourne - Migliore - Nagel]

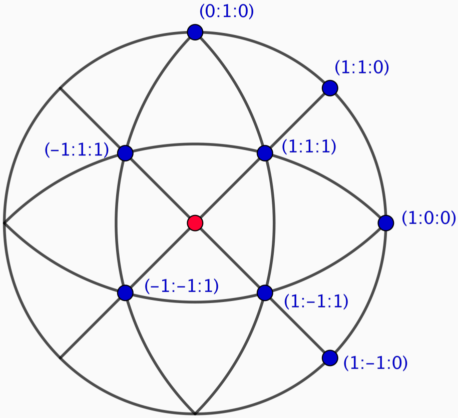
$$f_{\mathcal{A}} = (x)(y)(x-y)(x-y-z)(x-y+z)(x+y)(x+y-z)(x+y+z), \quad \deg(f_{\mathcal{A}}) = \#(\mathcal{A}) = 8$$

$$0 \longrightarrow \begin{matrix} S(-7-3) \\ \oplus \\ S(-7-4) \end{matrix} \longrightarrow S(-7)^3 \longrightarrow S \longrightarrow S/J_{\mathcal{A}} \longrightarrow 0$$

The splitting type of $\mathcal{A}$ is $(3, 4)$ $\implies$ there are no unexpected curves!

Example [Cook II - Harbourne - Migliore - Nagel]

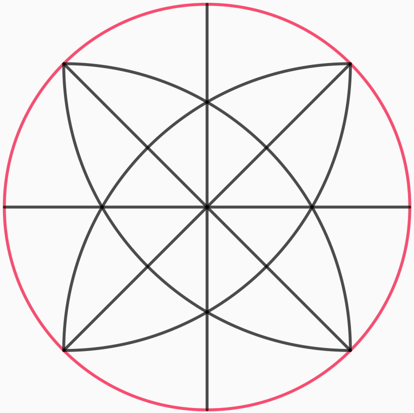
We add the point $(0:0:1)$



$$Z = \{(1:0:0),\ (0:1:0),\ (0:0:1),\ (1:-1:0),\ (-1:1:1),\ (1:-1:1),\ (1:1:0),\ (-1:-1:1),\ (1:1:1)\}.$$

Example [Cook II - Harbourne - Migliore - Nagel]

We add the line at infinity $z = 0$



$$f_{\mathcal{A}} = (x)(y)(z)\cdot$$
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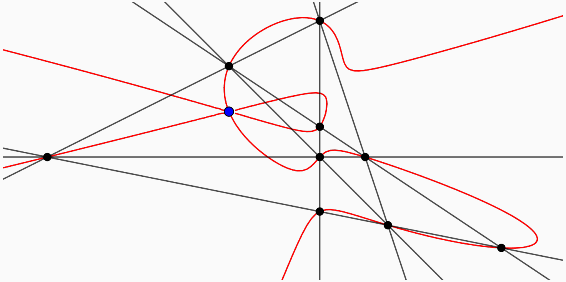
Example [Cook II - Harbourne - Migliore - Nagel]

$$f_{\mathcal{A}} = (x)(y)(z)(x-y)(x-y-z)(x-y+z)(x+y)(x+y-z)(x+y+z), \quad \deg(f_{\mathcal{A}}) = \#(\mathcal{A}) = 9$$

$$0 \longrightarrow \begin{matrix} S(-8-3) \\ \oplus \\ S(-8-5) \end{matrix} \longrightarrow S(-8)^3 \longrightarrow S \longrightarrow S/J_{\mathcal{A}} \longrightarrow 0$$

The splitting type of $\mathcal{A}$ is $(3, 5)$

Example [Cook II - Harbourne - Migliore - Nagel]



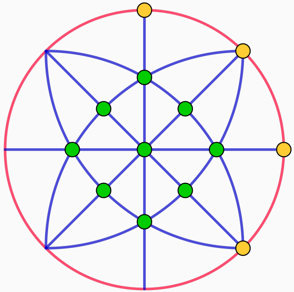
The splitting type of $\mathcal{A}$ is $(3, 5)$ $\Longrightarrow$ there is an unexpected quartic!

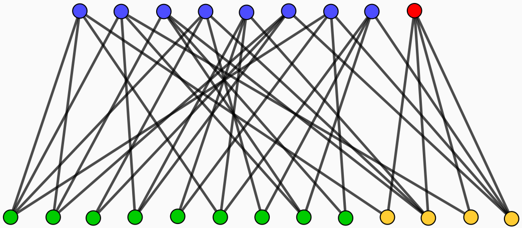
Motivations and Connections

Terao's Conjecture on freeness of line arrangements

Terao's Conjecture.

The freeness of a line arrangement depends only on its incidence lattice.





Terao's Conjecture on freeness of line arrangements

Proposition. [Cook II - Harbourne - Migliore - Nagel]

Let $\mathcal{A}$ and $\mathcal{A}'$ be two line arrangements with the same incidence lattice. Assume $\mathcal{A}$ is free with splitting type (a, b). Then one has:

1. $\mathcal{A}'$ is free if and only if $\mathcal{A}$ has the same splitting type as $\mathcal{A}'$;
2. If $\mathcal{A}'$ is not free, then its splitting type is $(a - s, b + s)$, for some positive integer s.

Corollary. [Cook II - Harbourne - Migliore - Nagel]

If the splitting type of a line arrangement is a combinatorial property, then Terao's conjecture is true.

Strong Lefschetz Property of Artinian algebras

Definition. An Artinian algebra $A = S/I$ satisfies the **Strong Lefschetz Property** (SLP) at **range** k and in **degree** d if, for a general linear form ℓ, the homomorphism $\times\ell^k : A_d \longrightarrow A_{d+k}$ has maximal rank, i.e., it is either surjective or injective.

Theorem. [Cook II - Harbourne - Migliore - Nagel] Let $\mathcal{A}$ be a line arrangement defined by $f = \ell_1 \cdots \ell_d$ and let Z be the dual configuration of points.

Then, the following are equivalent:

1. Z admits an unexpected curve of degree $j+1$
2. $S/(\ell_1^{j+1}, \dots, \ell_d^{j+1})$ fails SLP in range 2 and degree $j-1$.

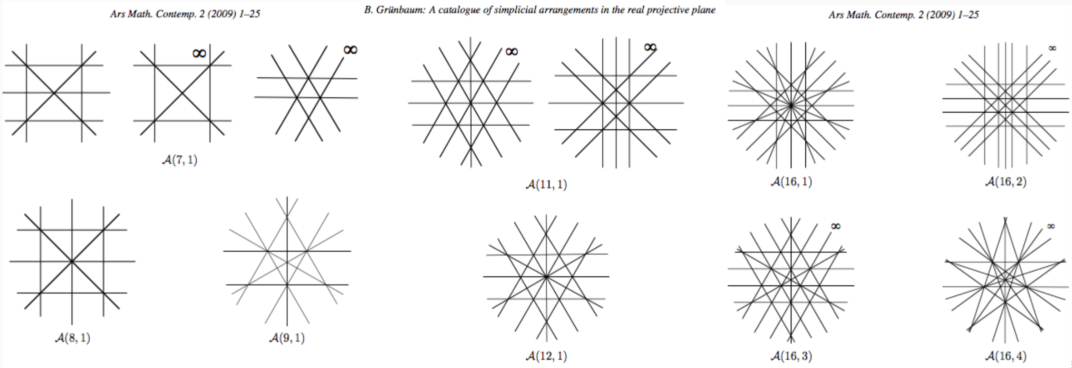
Key fact. [Emsalem - Iarrobino '95; Geramita '96] For any $[\ell_i] \in \mathbb{P}(S_1)$, denote $\wp_i = I([\ell_i])$.

$$\dim_{\mathbb{C}}[S/(\ell_1^{j+1}, \dots, \ell_d^{j+1}, \ell_{d+1}^2)]_{j+1} = \dim_{\mathbb{C}}[\wp_1 \cap \ldots \cap \wp_d \cap \wp_{d+1}^j]_{j+1}$$

New examples

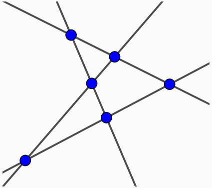
Simplicial line arrangements

Grünbaum, B. A catalogue of simplicial arrangements in the real projective plane,
Ars Mathematica Contemporanea 2 : 1–25, 2009.

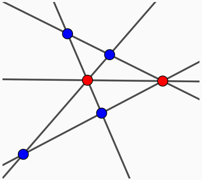


Supersolvable line arrangements

Definition.
A line arrangement is called **supersolvable** if there exists a **modular point**, i.e., $\exists P \in \mathrm{Sing}(\mathcal{A})$ such that $\forall Q \in \mathrm{Sing}(\mathcal{A}), \overline{PQ} \in \mathcal{A}$.



non-example



example

We call **multiplicity** of the arrangement $m = m(\mathcal{A}) = \max\{m(P) \ : \ P \in \mathrm{Sing}(\mathcal{A})\}$.

Supersolvable line arrangements

Lemma. Let $\mathcal{A}$ be a supersolvable line arrangement with $d = \#(\mathcal{A})$ and multiplicity m. Then, the splitting type of $\mathcal{A}$ is $(m-1, d-m)$.

Sketch of proof.

1. If $d - m = 0$, i.e., $\mathcal{A}$ is a "star", then prove it by induction on m.
2. If $d \geq m$, then prove it by induction on $d - m$.

In both steps, we use the following lemma.

Lemma. [Orlik-Terao] Let $\mathcal{A}$ be a line arrangement and $L \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} \setminus \{L\}$. If:

i) $\mathcal{A}'$ is free and has splitting type (a, b); ii) $\#\left(\mathrm{Sing}(\mathcal{A}) \cap L\right) = b + 1$ (or $a + 1$, respectively);

then, $\mathcal{A}$ is free with splitting type $(a+1, b)$ (or $(a, b+1)$, respectively).

Supersolvable line arrangements

Theorem. [Di Marca - Malara - O.]

Let $\mathcal{A}$ be a supersolvable line arrangement with $d = \#(\mathcal{A})$ and multiplicity m. Let Z be the dual configuration of points. Then,

$$Z \text{ admits unexpected curves if and only if } d > 2m.$$

In this case, Z admits unexpected curves of degree $j+1$ if and only if

$$m - 1 \leq j \leq d - m - 2.$$

Supersolvable line arrangements

Proof.

By Lemma, the splitting type of $\mathcal{A}$ is $(m-1, d-m)$.

Recall characterization by **[Cook II - Harbourne - Migliore - Nagel]**:

i. $2a+2 < \#(\mathcal{A})$ ii. there are no $a+1$ collinear points, i.e., $m(P) < a+1$, $\forall P \in \mathrm{Sing}(\mathcal{A})$

Now:

1. if $m-1 \leq d-m$, then

   i. $2(m-1)+2 < d \Leftrightarrow 2m < d$; ii. $\forall P \in \mathrm{Sing}(\mathcal{A})$. $m(P) \leq (m-1)+1 = m$ ✓

2. if $m-1 \geq d-m$, then

   i. $2(d-m)+2 < d \Leftrightarrow d < 2m-2$;

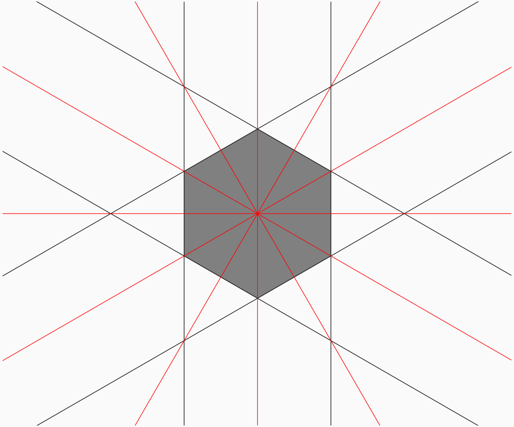
   ii. $\forall P \in \mathrm{Sing}(\mathcal{A})$. $m(P) \leq (d-m)+1 = m \Rightarrow m \leq d-m+1 \Rightarrow d \geq 2m-1$ $\to\leftarrow$

Polygonal arrangements

Hexagonal arrangement

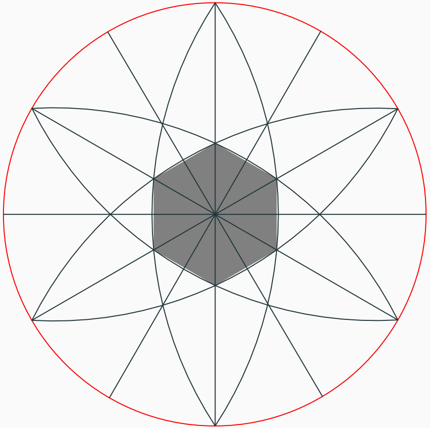
Polygonal arrangements

$\mathcal{P}_6$



Polygonal arrangements

$\overline{\mathcal{P}_6}$



Polygonal arrangements

Theorem. [Di Marca - Malara - O.]

Let N be a positive integer.

1. The configuration f points dual to $\mathcal{P}_N$ admits no unexpected curves.
2. If N is even, then the configuration of points dual to $\overline{\mathcal{P}_N}$ admits an unexpected curve of degree N.

Proof.

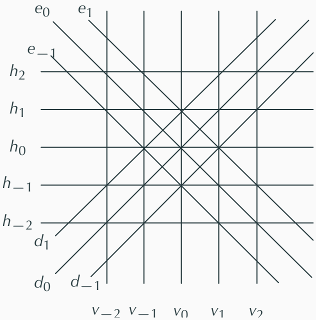
In both cases, $m(\mathcal{P}_N) = m(\overline{\mathcal{P}_N}) = N$.

1. $|\mathcal{P}_N| = 2N$, hence, the splitting type is $(N-1, N)$;
2. $|\overline{\mathcal{P}_N}| = 2N+1$, hence, the splitting type is $(N-1, N+1)$.

Tic-tac-toe arrangements

A **tic-tac-toe arrangement of type** (k, j), denoted $\mathcal{T}_k^j$, is the arrangement defined by:

1. v_i, $i = -k, \dots, k$: vertical lines $x = iz$;
2. h_i, $i = -k, \dots, k$: horizontal lines $y = iz$;
3. d_i, $i = -j, \dots, j$: the diagonals $x - y + iz = 0$;
4. e_i, $i = -j, \dots, j$:
   the anti-diagonals $x + y + iz = 0$.



The tic-tac-toe arrangement of type $(2, 1)$.

Tic-tac-toe arrangements

Theorem. [Di Marca - Malara - O.]

The complete tic-tac-toe arrangement $\mathcal{T}_k^j$ has an unexpected curve of degree $2(k+j+1)$.

Proof.

The splitting type is $(2k+2j+1, 2k+2j+3)$.

...more examples in the (coming soon) paper!

Tic-tac-toe arrangements

Theorem. [Di Marca - Malara - O.]

The complete tic-tac-toe arrangement $\overline{\mathcal{T}}^j_k$ has an unexpected curve of degree $2(k+j+1)$.

Proof.

The splitting type is $(2k+2j+1, 2k+2j+3)$.

...more examples in the (coming soon) paper!

Grazie per l'attenzione!

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COMING SOON

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