Waring loci and decompositions of low rank symmetric tensors

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joint works with E. Carlini – M.V. Catalisano, and B. Mourrain

Introduction

Waring decompositions and Waring ranks

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$.

DEFINITION

For $F \in S_d$, a **Waring decomposition** is an expression as

$$F = \sum_{i=1}^{r} L_i^d, \qquad \text{where } L_i \in S_1.$$

The **Waring rank** of F is the smallest length of such a decomposition – $\mathrm{rk}(F)$.

QUESTION

Given $F \in S_d$,

what is $\mathrm{rk}(F)$? How to construct a minimal Waring decomposition?

Waring decompositions and Waring ranks

- The rank of a generic form of fixed degree d and fixed number of variables $n+1$ is known (Alexander-Hirschowtiz Theorem, 1995);
- the rank of binary forms $(n = 1)$ is known (Sylvester's Algorithm);
- the rank of monomials is known (Carlini-Catalisano-Geramita, 2012);
- the rank of ternary cubics is know (classical, e.g., see Landsberg-Teitler);
- algorithms to compute minimal Waring decompositions and ranks have been presented (Brachat-Comon-Mourrain-Tsigaridas, Oeding-Ottaviani, Bernardi-Gimigliano-Idà,...
- ...

Waring and forbidden loci

DEFINITION (Carlini-Catalisano-O.)
For $F \in S_d$, the **Waring locus** of F is the locus of linear forms that can appear in a minimal Waring decomposition of F, i.e.,

$$\mathcal{W}_F := \left\{ [L] \in \mathbb{P}(S_1) \mid \exists L_2, \ldots, L_r \in S_1 \text{ s.t. } F \in \langle L^d, L_2^d, \ldots, L_r^d \rangle \text{ and } r = \mathrm{rk}(F) \right\}.$$

The **forbidden locus** of F is the complement $\mathcal{F}_F := \mathbb{P}(S_1) \setminus \mathcal{W}_F$.

Waring and forbidden loci

"Waring loci should be considered in the space of linear forms with the essential number of variables"

DEFINITION

The **essential number** of variables of $F \in \mathbb{C}[x_0, \ldots, x_n]$ is the smallest m such that $F \in \mathbb{C}[z_1, \ldots, z_m]$, where $z_i(x_0, \ldots, x_n)$'s are linear forms.

Waring and forbidden loci

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LEMMA

Let $F \in S_d$ and assume that $F \in \mathbb{C}[z_1, \ldots, z_m]$, where $z_i(x_0, \ldots, x_n)$'s are linear forms. If $F = \sum_{i=1}^{\mathrm{rk}(F)} L_i^d$, then $L_i \in \mathbb{C}[z_1, \ldots, z_m]$.

Waring and forbidden loci

"Waring loci should be considered in the space of linear forms with the essential number of variables"

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The **essential number** of variables of $F \in \mathbb{C}[x_0, \ldots, x_n]$ is the smallest m such that $F \in \mathbb{C}[z_1, \ldots, z_m]$, where $z_i(x_0, \ldots, x_n)$'s are linear forms.

LEMMA

Let $F \in S_d$ and assume that $F \in \mathbb{C}[z_1, \ldots, z_m]$, where $z_i(x_0, \ldots, x_n)$'s are linear forms. If $F = \sum_{i=1}^{\mathrm{rk}(F)} L_i^d$, then $L_i \in \mathbb{C}[z_1, \ldots, z_m]$.

Hence, if $H = \langle [z_1], \ldots, [z_m] \rangle \simeq \mathbb{P}^{m-1} \subset \mathbb{P}(S_1)$, then $\mathcal{W}_F \subset H$.

Waring and forbidden loci

"Waring loci can be used to construct iteratively minimal Waring decompositions"

Indeed, if $[L] \in \mathcal{W}_F$, then there exists $\lambda \in \mathbb{C}$ such that

$$\mathrm{rk}(F + \lambda L^d) = \mathrm{rk}(F) - 1.$$

In particular:

THEOREM (Mourrain - O.)

If $F \in S_d$ and $\mathrm{rk}(F) > r^\circ(d, n)$, where $r^\circ(d, n)$ is the Waring rank of a generic form in S_d, then $\mathcal{W}_F$ is dense in $\mathbb{P}(S_1)$.

Waring and forbidden loci

"forbidden loci might be used to find forms of high rank"

Indeed, if $[L] \in \mathcal{F}_F$, then, for any $\lambda \in \mathbb{C}$,

$$\mathrm{rk}(F + \lambda L^d) \geq \mathrm{rk}(F).$$

In general, we do not know the maximal Waring rank of forms of fixed degree and we are not even aware of forms of rank higher than the generic one.

Waring and forbidden loci

"Waring and forbidden loci are constructible, but..."

...as we will see, they can be open, closed or neither one or the other.

Moreover, we do not know if the forbidden locus is always non-empty (in all our examples, it does).

Waring loci via Apolarity Theory

Apolar action

Let $S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} S_d$ and $R = \mathbb{C}[y_0, \ldots, y_n] = \bigoplus_{d \geq 0} R_d$.

Consider the **apolar action** of R on S as partial derivatives, i.e.,

$$\begin{aligned} \circ : R \times S &\longrightarrow S \\ (G, H) &\mapsto G \circ F := G(\partial_0, \ldots, \partial_n)F \end{aligned}$$

DEFINITION

Given $F \in S_d$, we define the **apolar ideal** of F as

$$F^{\perp} = \{G \in R \mid G \circ F = 0\} \subset R.$$

Apolarity Lemma I

LEMMA (Apolarity Lemma)

The following are equivalent:

- $F \in \langle L_1^d, \ldots, L_r^d \rangle$;
- $F^\perp \supset I(\mathbb{X})$, where $\mathbb{X} = \{[L_1], \ldots, [L_r]\} \subset \mathbb{P}(S_1)$.

DEFINITION

A set of points $\mathbb{X}$ whose ideal $I(\mathbb{X})$ is contained in the apolar ideal of F is said to be **apolar** to F.

Apolarity Lemma II

The definition of the Waring and forbidden loci can be rephrased as follows.

DEFINITION

For $F \in S_d$, the **Waring locus** of F is the set of points which can be completed to a minimal set of points apolar to F, i.e.,

$$\mathcal{W}_F := \left\{ P \in \mathbb{P}^n \mid \exists \mathbb{X} = \{P, P_2, \ldots, P_r\} \text{ s.t. } F^\perp \supset I(\mathbb{X}) \text{ and } r = \mathrm{rk}(F) \right\}.$$

The **forbidden locus** of F is the complement $\mathcal{F}_F := \mathbb{P}^n \smallsetminus \mathcal{W}_F$.

Waring loci of binary forms I

The apolar ideal of a binary form $F \in S_d = \mathbb{C}[x_0, x_1]_d$ is of the form

$$F^\perp = (G_1, G_2), \quad \text{where } \deg(G_1) + \deg(G_2) = d + 2.$$

The ideal of a reduced set of points in $\mathbb{P}^1$ is a principal ideal with a square-free generator.

Sylvester's Algorithm.

- Compute $F^\perp$, say $\deg(G_1) \leq \deg(G_2)$;
- if G_1 is square-free, then $\mathrm{rk}(F) = \deg(G_1)$;
- otherwise, there is a square-free polynomial $G_1 M + G_2$ and $\mathrm{rk}(F) = \deg(G_2)$.

Waring loci of binary forms II

THEOREM (Carlini-Catalisano-O.)

Let $F \in S_d = \mathbb{C}[x_0, x_1]_d$ and $F^\perp = (G_1, G_2)$, with $\deg(G_1) \leq \deg(G_2)$. Then:

- if $\mathrm{rk}(F) < \left\lceil \frac{d+1}{2} \right\rceil$, then $\mathcal{W}_F = V(G_1)$;
- if $\mathrm{rk}(F) > \left\lceil \frac{d+1}{2} \right\rceil$, then $\mathcal{F}_F = V(G_1)$;
- if $\mathrm{rk}(F) = \left\lceil \frac{d+1}{2} \right\rceil$, then: if d is even, then $\mathcal{F}_F$ is finite and not empty; if d is odd, then $\mathcal{W}_F = V(G_1)$.

COROLLARY

Let $F \in S_d$ such that $\mathrm{rk}(F) = r \geq \left\lceil \frac{d+1}{2} \right\rceil$. For any $[L_1], \ldots, [L_s] \in \mathcal{W}_F$, for $s = r - \left\lceil \frac{d+1}{2} \right\rceil$, there exists $L_{s+1}, \ldots, L_r \in S_1$ such that $F \in \langle L_1, \ldots, L_r \rangle$.

Waring loci of monomials I

Let $M = x_0^{a_0} \cdots x_n^{a_n}$, with $0 < a_0 = a_1 = \ldots = a_m < a_{m+1} \leq \ldots \leq a_n$.

Recall that, by [Carlini-Catalisano-Geramita], we have $\mathrm{rk}(M) = \frac{1}{a_0+1} \prod_{i=0}^{n} (a_i+1)$.

THEOREM (Carlini-Catalisano-O.)

In the same notation as above, $\mathcal{F}_M = V(y_0 \cdots y_m)$.

Proof.

Recall that $M^\perp = (y_0^{a_0+1}, \ldots, y_n^{a_n+1})$. Let $P = [p_0 : \ldots : p_n] \notin V(y_0 \cdots y_m)$; assume $p_0 = 1$. Then, let

$$H_i = y_i^{a_i+1} - p_i^{a_i+1} y_0^{a_i+1}, \text{ if } p_i \neq 0, \text{ and } H_i = y_i^{a_i+1} - y_i y_0^{a_i}, \text{ if } p_i = 0.$$

Then, $(H_1, \ldots, H_n)$ defines a minimal set of points apolar to M and containing P.

Waring loci of monomials II

Viceversa, let $\mathbb{X}$ be a minimal set of points apolar to M, i.e., $I(\mathbb{X}) \subset F^{\perp}$. Then, let $\mathbb{X}' = \mathbb{X} \smallsetminus \{\mathbb{X} \cap \{x_0 = 0\}\}$, i.e., $I(\mathbb{X}') = I(\mathbb{X}) : (x_0)$.

Now, recall that:

- the Hilbert function of the quotient ring of a set of reduced points is increasing, until it stabilizes to the cardinality of $\mathbb{X}$; hence,

$$|\mathbb{X}'| = HF_{R/I(\mathbb{X}')}(d), \qquad \text{for } d \gg 0;$$

- since x_0 is a non-zero divisor of $I(\mathbb{X}')$, the Hilbert function of $R/(I(\mathbb{X}') + (y_0))$ corresponds to the first difference of the Hilbert function of $S/I(\mathbb{X}')$; hence,

$$HF_{R/I(\mathbb{X}')}(d) = \sum_{i=0}^{\infty} HF_{R/(I(\mathbb{X}')+(y_0))}(i), \qquad \text{for } d \gg 0;$$

Waring loci of monomials III

- since $I(\mathbb{X}) \subset F^{\perp}$, we have that

$$I(\mathbb{X}) : (y_0) + (y_0) \subset F^{\perp} : (y_0) + (y_0) = (y_0, y_1^{a_1+1}, \ldots, y_n^{a_n+1});$$

In conclusion,

$$|\mathbb{X}| \geq |\mathbb{X}'| \geq \sum_{i=0}^{\deg(F)} HF_{R/(y_0, y_1^{a_1+1}, \ldots, y_n^{a_n+1})}(i) = \prod_{i=1}^{n}(a_i + 1) = |\mathbb{X}|,$$

i.e., $\mathbb{X}$ does not have points on $\{x_0 = 0\}$. Similarly for any $\{x_i = 0\}$, with $i = 1, \ldots, m$.

□

Waring loci of plane cubics I

(e.g., see [Landsberg-Teitler])

Type	Description	Normal form	Waring rank
(1)	triple line	x_0^3	1
(2)	three concurrent lines	$x_0x_1(x_0+x_1)$	2
(3)	double line + line	$x_0^2x_1$	3
(4)	smooth	$x_0^3+x_1^3+x_2^3$	3
(5)	three non-concurrent lines	$x_0x_1x_2$	4
(6)	line + conic (meeting transversally)	$x_0(x_1x_2+x_0^2)$	4
(7)	nodal	$x_0x_1x_2-(x_1+x_2)^3$	4
(8)	cusp	$x_0^3-x_1^2x_2$	4
(9)	general smooth ($a^3 \neq -27, 0, 6^3$)	$x_0^3+x_1^3+x_2^3+ax_0x_1x_2$	4
(10)	line + tangent conic	$x_0(x_0x_1+x_2^2)$	5

Waring loci of plane cubics II

LEMMA (Carlini-Catalisano-O.)

Let $F \in \mathbb{C}[x_0, x_1, x_2]_3$ of rank 4. Then, F is a cusp if and only if has a minimal apolar set of points with three collinear points.

Proof.

Assume $F = x_0^3 - x_1^2 x_2$. Then, consider $x_0^3 + \sum_{i=1}^3 L_i(x_1, x_2)^3$.

Viceversa, we may assume that F has a minimal set of point given by the union of $[x_0]$ and three collinear points on $\{y_0 = 0\}$. I.e.,

$$(y_0) \cap (y_1, y_2) = (y_0 y_1, y_0 y_2) \subset F^{\perp}.$$

In particular, $F = x_0^3 + G(x_1, x_2)$, where G is a cubic of rank 3, i.e., $G = L^2 M$, for some linear forms L, M. □

Waring loci of plane cubics III

"The Waring locus might be neither open nor closed"

THEOREM (Carlini-Catalisano-O.)

Let $F = x_0^3 - x_1^2 x_2$ be a plane cubic cusp. Then,

$$\mathcal{W}_F = [x_0] \amalg \mathcal{W}_{x_1^2 x_2} = [x_0] \amalg \left(\mathbb{P}(\mathbb{C}[x_1, x_2]_1) \smallsetminus [x_2] \right).$$

Waring loci of plane cubics IV

As regards plane cubics of rank 4 which are not cusps, we consider the following strategy:

- let $\mathcal{L} = (F^{\perp})_2 = \langle C_1, C_2, C_3 \rangle$: minimal set of points are given by complete intersections in $\mathcal{L}$;
- for any $P \in \mathbb{P}^2$, consider $\mathcal{L}(-P)$: if it has 4 base points, then $P \in \mathcal{W}_F$; otherwise, $P \in \mathcal{F}_F$;
- the ideal of four points which are a complete intersection contains three reducible conics: inside $\mathbb{P}(\mathcal{L})$ let Δ be the cubic of reducible conics, if the line $\mathcal{L}(-P)$ is a proper tri-secant line, then $P \in \mathcal{W}_F$, otherwise $P \in \mathcal{F}_F$.

EXCELENCIA MARÍA DE MAEZTU

Waring loci of plane cubics V

Hence, if F is a rank 4 ternary cubic which is not a cusp, consider

$$\mathcal{L} = (F^{\perp})_2 = \langle C_1, C_2, C_3 \rangle.$$

Let $\check{\Delta} \subset \check{\mathbb{P}}(\mathcal{L})$ be the dual curve of lines which do not intersect properly the cubic of reducible conics $\Delta \subset \mathbb{P}(\mathcal{L})$ and consider

$$\varphi : \mathbb{P}^2 \longrightarrow \check{\mathbb{P}}(\mathcal{L}), \quad \varphi((a:b:c)) = (C_1(a,b,c) : C_2(a,b,c) : C_3(a,b,c)).$$

Then,

$$\mathcal{F}_F = \varphi^{-1}(\check{\Delta}).$$

Waring loci of plane cubics VI

"The forbidden locus can be very small"

THEOREM (Carlini-Catalisano-O.)

Let $F = x_0(x_0x_1 + x_2^2$ be a plane cubic of rank 5. Then, $\mathcal{F}_F = \{[x_0]\} \subset \mathbb{P}(S_1)$.

Waring loci and Symmetric Strassen's Conjecture I

CONJECTURE (Symmetric Strassen's Conjecture)

Let $F_i \in \mathbb{C}[x_{i,0}, \ldots, x_{i,n_i}]_d$, for $i = 1, \ldots, s$. Then,

$$\mathrm{rk}\left(\sum_{i=1}^{s} F_i\right) = \sum_{i=1}^{s} \mathrm{rk}(F_i).$$

THEOREM (Carlini-Catalisano-Chiantini-Geramita-Woo)

The Symmetric Strassen's Conjecture holds if the F_i's are:

- monomials;
- forms in one or two variables;
- $x_0^a(x_1^b + \ldots + x_n^b)$, for $a + 1 \geq b$;
- $x_0^a(x_0^b + \ldots + x_n^b)$, for $a + 1 \geq b$;
- ...

Waring loci and Symmetric Strassen's Conjecture II

CONJECTURE (Strong Symmetric Strassen's Conjecture)

Let $F_i \in \mathbb{C}[x_{i,0}, \ldots, x_{i,n_i}]_d$, for $i = 1, \ldots, s$. Then, every minimal decomposition of $F = \sum_i F_i$ is a sum of minimal decompositions of the F_i's.

REMARK

Let $F_1 \in \mathbb{C}[x_0, \ldots, x_n]$ and $F_2 \in [y_0, \ldots, y_m]$. Then,

$$\mathcal{W}_{F_1} \subset \{y_i = 0\} \simeq \mathbb{P}^n \subset \mathbb{P}^{n+m+1} \quad \text{and} \quad \mathcal{W}_{F_2} \subset \{x_i = 0\} \simeq \mathbb{P}^m \subset \mathbb{P}^{n+m+1}.$$

CONJECTURE (Disjoint Waring loci Conjecture)

Let $F_i \in \mathbb{C}[x_{i,0}, \ldots, x_{i,n_i}]_d$, for $i = 1, \ldots, s$. Then, $\mathcal{W}_F = \coprod_i \mathcal{W}_{F_i}$.

Waring loci and Symmetric Strassen's Conjecture III

THEOREM (Carlini-Catalisano-O.)

Disjoint Waring loci Conjecture $\Longleftrightarrow$ Strong Symmetric Strassen's Conjecture

$\searrow \quad \swarrow$

Symmetric Strassen's Conjecture

Decompositions of low rank forms via Waring loci

Equations for Waring locus via regularity I

Given a set of points $\mathbb{X}$, let $\rho(\mathbb{X})$ be the **regularity index** of $\mathbb{X}$, i.e.,

$$\min\{i \mid HF_{S/I(\mathbb{X})}(i) = |\mathbb{X}|\}.$$

LEMMA (Mourrain-O.)

Let $F \in S_d$ and let $\mathbb{X}$ be a minimal set of points apolar to F. Assume $d \geq \rho(\mathbb{X})$. Then,

$$(I(\mathbb{X}))_i = F_i^{\perp}, \quad \forall 0 \leq i \leq d - \rho(\mathbb{X}).$$

Idea of the proof.

$$\mathrm{cat}_i(F) = \mathrm{van}_{d-i}(\mathbb{X})^T \cdot D \cdot \mathrm{van}_i(\mathbb{X}), \quad D \text{ is a diagonal matrix.}$$

For $d - i \geq \rho(\mathbb{X})$, we have that $\mathrm{van}_{d-i}(\mathbb{X})$ is injective. Hence, the kernel of $\mathrm{cat}_i(F)$, i.e., $F_i^{\perp}$, coincides with the kernel of $\mathrm{van}_i(\mathbb{X})$, i.e., $I(\mathbb{X})_i$.

Equations for Waring locus via regularity II

THEOREM (Mourrain-O.)

Let $F \in S_d$ and let $\mathbb{X}$ be a minimal set of points apolar to F.
Assume $d \geq 2\rho(\mathbb{X}) + 1$. Then,

$$I(\mathbb{X}) = (F^{\perp})_{\leq \rho(\mathbb{X})+1}.$$

Moreover, $\mathbb{X}$ is the unique minimal set of points apolar to F.

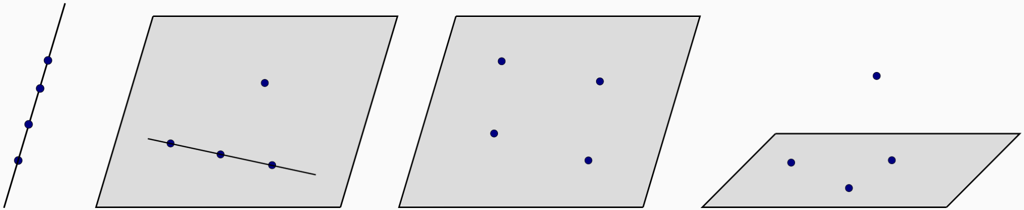
Equations for Waring locus via regularity III

QUESTION

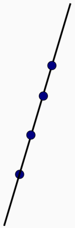
Let $\mathbb{X}$ and $\mathbb{X}'$ two minimal set of points apolar to F.

What algebraic properties they share? E.g., do they have the same regularity?

Decompositions of symmetric tensors of rank 4 I

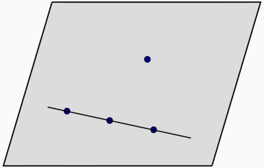


Decompositions of symmetric tensors of rank 4 II



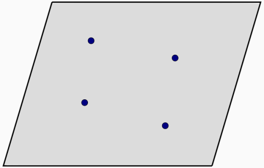
- F has two essential variables, i.e., $\mathrm{HF}_{R/F^\perp}(1) = 2$.
- $F^\perp = (L_1, \ldots, L_{n-1}, G_1, G_2)$;
- apply Sylvester's Algorithm:
  - if $d = 4, 5, 6$, then $I(\mathbb{X}) = (L_1, \ldots, L_{n-1}, G_1 H + \alpha G_2)$;
  - if $d \geq 7$, then $I(\mathbb{X}) = (L_1, \ldots, L_{n-1}, G_1)$.

Decompositions of symmetric tensors of rank 4 III

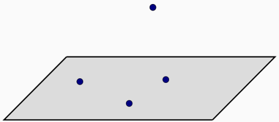


- F has three essential variables, i.e., $\mathrm{HF}_{R/F^\perp}(1) = 3$.
- if $d = 3$, then $V(F^\perp_2) = P + D$, where P is reduced and D is 0-dimensional of length 2:
  any minimal set of points is $P \cup \mathbb{X}'$, where $\mathbb{X}' \subset \langle D \rangle \simeq \mathbb{P}^1$.
- if $d = 4$, then $V(F^\perp_2) = P \cup \ell$, where P is reduced and ℓ is a line:
  any minimal set of points is $P \cup \mathbb{X}'$, where $\mathbb{X}' \subset \ell$.
- if $d \geq 5$, since $\rho(\mathbb{X}) = 2$, then $I(\mathbb{X}) = F^\perp_{\leq 3}$.

Decompositions of symmetric tensors of rank 4 IV

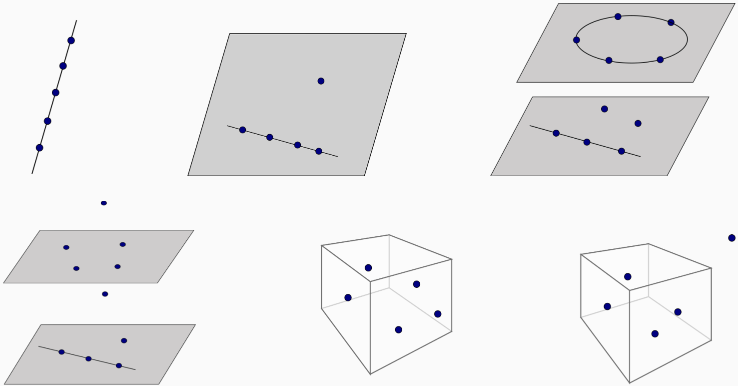


- F has three essential variables, i.e., $HF_{R/F^\perp}(1) = 3$.
- if $d = 3$, then $V(F_2^\perp) = \emptyset$ and $\mathcal{W}_F$ is dense; hence, start from a generic linear form and reduce the rank;
- if $d \geq 4$, then $I(\mathbb{X}) = F_2^\perp$.



- F has four essential variables, i.e., $HF_{R/F^\perp}(1) = 4$.
- $I(\mathbb{X}) = F_2^\perp$.

Decompositions of symmetric tensors of rank 5



Decompositions of symmetric tensors of rank ≤ 5

THEOREM (Mourrain-O.)

	Hilbert Sequence	Extra Condition	Algorithm to find a minimal apolar set
(1)	$[1^\bullet]$		$\mathrm{rk}(f) = 1$ *and* $(f_1^\perp)$ *defines the point apolar to* f
(2)	$[1, 2, *, 2, 1]$		f *has two essential variables and Sylvester algorithm is applied:* (i) *if* $f_{l(f)}^\perp$ *defines a set of* $l(f)$ *reduced points, then* $\mathrm{rk}(f) = l(f)$; (ii) *otherwise,* $\mathrm{rk}(f) = d + 2 - l(f)$ *and a minimal apolar set is given by the principal ideal generated by a generic form* $g \in f_{d+2-l(f)}^\perp$
(3)	$[1, 3, 3, 1]$	$Z(f_2^\perp) = \emptyset$	*a generic pair of conics* q_1, q_2 *of* $f_2^\perp$ *defines 4 points and* $\mathrm{rk}(f) = 4$
(4)	$[1, 3, 3, 1]$	$Z(f_2^\perp) = P \cup D$, P *is simple point* D *connected, 0-dim* $\deg(D) = 2$	$\mathrm{rk}(f) = 4$ *and* P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f' = f - c\ell_P^3$ *has two essential variables and we apply Sylvester algorithm to* f' *as in (2)*
(5)	$[1, 3, 3, 1]$	$Z(f_2^\perp) = D$ D *connected, 0-dim* $\deg(D) = 3$	$\mathrm{rk}(f) = 5$ *and, for a generic* P *and a generic* $c \neq 0$ *such that* $f' = f + c\ell_P^3$ *is a ternary cubic of rank 4 and we apply (4) to* f'
(6)	$[1, 3, 3^\bullet, 3, 1]$	$Z(f_2^\perp) = \{P_1, P_2, P_3\}$ P_i*'s are simple points*	$\mathrm{rk}(f) = 3$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(7)	$[1, 3, *, 3, 1]$	$Z(f_2^\perp) = P \cup L$ P *is simple point* L *is line,* $P \notin L$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f' = f - c\ell_P^d$ *has two essential variables and we apply Sylvester algorithm to* f' *as in (2)*
(8)	$[1, 3, 4^\bullet, 3, 1]$	$Z(f_2^\perp) = \{P_1, \ldots, P_4\}$ P_i*'s are simple points*	$\mathrm{rk}(f) = 4$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(9)	$[1, 3, 5, 3, 1]$	$Z(f_2^\perp) = C$ C *is irreducible quadric*	*let* P *be a generic point on* C *and* c *be a scalar such that* $f' = f - c\ell_P^4$ *has* $h_{f'}(2) = 4$. (i) *if* $Z((f')_2^\perp) = \{P_1, \ldots, P_4\}$ *is a set of 4 reduced points, then,* $\mathrm{rk}(f) = 5$, *and a minimal set apolar to* f *is* $\{P, P_1, \ldots, P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f) > 5$
(10)	$[1, 3, 5, 3, 1]$	$Z(f_2^\perp) = L_1 \cup L_2$ L_i*'are distinct lines*	*let* P_i *be a generic point on* L_i, *for* $i = 1, 2$, *respectively and* c_i *be a scalar such that* $f_i = f - c_i\ell_{P_i}^4$ *has* $h_{f_i}(2) = 4$, *for* $i = 1, 2$: (i) *if* $Z((f_i^\perp)_2) = \{P_1, \ldots, P_4\}$, *for either* $i = 1$ *or* $i = 2$, *then,* $\mathrm{rk}(f) = 5$, *and a minimal apolar set of* f *is* $\{P, P_1, \ldots, P_4\}$; (ii) *otherwise,* $\mathrm{rk}(f) > 5$
(11)	$[1, 3, 5, 5^\bullet, 3, 1]$	$Z(f_3^\perp) = \{P_1, \ldots, P_5\}$ P_i*'s are reduced points*	$\mathrm{rk}(f) = 5$ *and the unique minimal apolar set is* $Z(f_3^\perp)$
(12)	$[1, 4, 4, 1]$	$Z(f_2^\perp) = P \cup H$ P *is a reduced point* H *is a plane,* $P \notin H$	P *is a point of any minimal apolar set; then, we find the scalar* c *such that* $f' = f - c\ell_P^3$ *has three essential variables and we apply (3) or (4) to* f'
(13)	$[1, 4, 5^\bullet, 4, 1]$	$Z(f_2^\perp) = \{P_1, \ldots, P_5\}$	$\mathrm{rk}(f) = 5$ *and the unique minimal apolar set is* $Z(f_2^\perp)$
(14)	$[1, 5, 5^\bullet, 5, 1]$	$Z(f_2^\perp) = \{P_1, \ldots, P_5\}$	$\mathrm{rk}(f) = 5$ *and the unique minimal apolar set is* $Z(f_2^\perp)$

Grazie Mille!

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